Interest-Rate Models: Course Notes

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Spot-Rate Models

- Normal Rate (Gaussian) Models
  - Vasicek (1977)
  - Hull and White (1994)

- Lognormal Models
  - Black and Karasinski (1991) (BK)
  - Peterson, Stapleton and Subrahmanyam (2003), 2-factor BK

- Spot-rate Models
  - Assume a process for the spot short rate
  - Derive bond prices, given the spot rate process
  - Can be used to price derivatives (caps, swaptions)
References


Lecture 1: The Gaussian Model

- Vasicek-Hull and White type model
- Assume that the short ($\delta$- period) rate follows a normal distribution, mean-reverting process
- Discrete time process
- Under risk-neutral measure
Geometric Progressions

1. 

\[ S_1 = 1 + (1 - k) + (1 - k)^2 + \ldots + (1 - k)^{n-1} \]

\[ S_1 = \frac{1 - (1 - k)^n}{k} \]

2. 

\[ S_2 = 1 + (1 - k)^2 + (1 - k)^4 + \ldots + (1 - k)^{2(n-1)} \]

\[ S_2 = \frac{1 - (1 - k)^{2n}}{1 - (1 - k)^2} \]
Geometric Progressions

3.

\[ S_3 = n + (n - 1)(1 - k) + (n - 2)(1 - k)^2 + \ldots + 2(1 - k)^{(n-2)} + (1 - k)^{(n-1)} \]

\[ S_3 = \frac{1}{k} \left\{ n - (1 - k) \left[ \frac{1 - (1 - k)^n}{k} \right] \right\} \]

4.

\[ S_4 = n + (n - 1)(1 - k)^2 + (n - 2)(1 - k)^4 + \ldots + 2(1 - k)^{2(n-2)} + (1 - k)^{2(n-1)} \]

\[ S_4 = \frac{n - (1 - k)^2 \left[ \frac{1 - (1 - k)^{2n}}{1 - (1 - k)^2} \right]}{1 - (1 - k)^2} \]
Short-Rate Model

\[ r_{t+1} - r_t = k(a - r_t) + \varepsilon_{t+1} \]

short rate of interest, \( r_t \)
long-term mean of short rate, \( a \)
rate of mean reversion, \( k, \ 0 < k < 1 \)
\( \varepsilon_{t+1} \), drawing from a normal distribution, \( E_t(\varepsilon_{t+1}) = 0, \ var_t(\varepsilon_{t+1}) = \sigma^2 \).

Hence,

\[ r_{t+1} = ka + (1 - k)r_t + \varepsilon_{t+1} \ (1) \]

Equation (1) holds for all \( t \)
Short-Rate Model

Example: Short rate: 3-month Libor

$t = 0$

short rate, $r_0 = 0.05$

long term mean, $a = 0.06$

mean reversion, $k = 0.25$

variance, $\sigma^2 = 0.0004$

$$r_1 = 0.25(0.06) + 0.75(0.05) + \varepsilon_1$$
Mean and Variance of the Short-Rate

\[ r_{t+1} = ka + (1 - k)r_t + \varepsilon_{t+1} \quad (1) \]

\[ r_{t+2} = ka + (1 - k)r_{t+1} + \varepsilon_{t+2} \quad (2) \]

Substitute (1) in (2)

\[ r_{t+2} = ka + (1 - k)ka + (1 - k)^2 r_t + (1 - k)\varepsilon_{t+1} + \varepsilon_{t+2} \]

Mean of \( r_{t+2} \):

\[ E_t(r_{t+2}) = ka + (1 - k)ka + (1 - k)^2 r_t \quad (3) \]

Variance of \( r_{t+2} \):

\[ var_t(r_{t+2}) = (1 - k)^2 var(\varepsilon_{t+1}) + var(\varepsilon_{t+2}) \quad (4) \]

Exercise: Find the mean and variance of \( r_{t+3} \)
Mean and Variance of $r_{t+n}$

$$r_{t+n} = ka + (1 - k)ka + (1 - k)^2ka + ... + (1 - k)^{n-1}ka$$
$$+ (1 - k)^n r_t$$
$$+ (1 - k)^{n-1} \varepsilon_{t+1} + (1 - k)^{n-2} \varepsilon_{t+2} + ... + \varepsilon_{t+n}$$

Hence

$$E_t(r_{t+n}) = ka + (1 - k)ka + (1 - k)^2ka + ...$$
$$+ (1 - k)^{n-1}ka + (1 - k)^n r_t$$
$$= ka[1 + (1 - k) + (1 - k)^2 + ... + (1 - k)^{n-1}]$$

Variance of $r_{t+n}$:

$$\text{var}_t(r_{t+n}) = (1 - k)^{2(n-1)} \text{var}(\varepsilon_{t+1}) + (1 - k)^{2(n-2)} \text{var}(\varepsilon_{t+2}) + ... + \text{var}(\varepsilon_{t+n})$$

and if

$$\text{var}(\varepsilon_{t+1}) = \text{var}(\varepsilon_{t+2}) = ... = \text{var}(\varepsilon_{t+n}) = \sigma^2$$

$$\text{var}_t(r_{t+n}) = \sigma^2[(1-k)^{2(n-1)}+(1-k)^{2(n-2)}+...+1]$$
Mean and Variance of $r_{t+n}$

Using the results from Geometric Progression:

Mean of $r_{t+n}$:

$$E_t(r_{t+n}) = a[1 - (1 - k)^n] + (1 - k)^n r_t$$

Variance of $r_{t+n}$:

$$var_t(r_{t+n}) = \sigma^2 \left[ \frac{1 - (1 - k)^{2n}}{1 - (1 - k)^2} \right]$$

Annualised st.dev.

$$std_t(r_{t+n}) = \sqrt{var_t(r_{t+n})/n}$$
Long-Bond Prices and Yields

Some key results:

1. Mean of a lognormal variable:
   Let $x$ be normally distributed, with $(\mu, \sigma^2)$ then
   \[
   E(e^{bx}) = e^{b\mu + 0.5(b^2\sigma^2)}
   \]

2. Forward price of a bond:
   The forward price of a 1-period bond is
   \[
   B_{t, t+\tau, t+\tau+1} = E_t[B_{t+\tau, t+\tau+1}]
   \]

3. The price of a long-term bond
   \[
   B_{t, t+n} = B_{t, t+1}B_{t+1, t+2}\ldots B_{t+\tau, t+\tau+1}\ldots B_{t+n-1, t+n}
   \]
Long-Bond Prices and Yields

\[ B_{t,t+\tau,t+\tau+1} = E_t[e^{-r_{t+\tau}}] \]

\[ B_{t,t+\tau,t+\tau+1} = e^{-\mu_{t+\tau} + \frac{\text{var}_t(r_{t+\tau})}{2}} \]

\[ \mu_{t+\tau} = E_t[\ln r_{t+\tau}] \]

Substitute in the equation for \( B_{t,t+n} \) to obtain the long bond price.

Bond yields:
Define the yield by \( y_{t+n} \) in

\[ B_{t,t+n} = e^{-y_{t+n}n} \]

and solve for

\[ y_{t+n} = \frac{-\ln (B_{t,t+n})}{n} \]
Long-Bond Prices and Yields

The price of a long-term bond

\[
B_{t,t+3} = B_{t,t+1} B_{t,t+1,t+2} B_{t,t+2,t+3} \\
= e^{-r_t} E_t[e^{-r_{t+1}]} E_t[e^{-r_{t+2}}] \\
= e^{-r_t} e^{-\mu_{t+1} + \frac{\text{var}(r_{t+1})}{2}} e^{-\mu_{t+2} + \frac{\text{var}(r_{t+2})}{2}} \\

B_{t,t+3} = e^{-[r_t + ka + (1-k)r_t + ka + (1-k)ka + (1-k)^2 r_t]} \\
\cdot e^{\frac{1}{2}[2\sigma^2 + (1-k)^2 \sigma^2]} \\
= e^{-[r_t[1+(1-k)+(1-k)^2]} e^{-[2ka+(1-k)ka]} \\
\cdot e^{\frac{1}{2}[2\sigma^2 + (1-k)^2 \sigma^2]}
Long-Bond Prices and Yields

\[ B_{t,t+n+1} = e^{-r_t[1+(1-k)+\ldots+(1-k)^n]} \]
\[ \cdot e^{-\left[nka+(n-1)(1-k)ka+\ldots+(1-k)^{n-1}ka\right]} \]
\[ \cdot e^{\frac{1}{2}\left[n\sigma^2+(n-1)(1-k)^2\sigma^2+\ldots+(1-k)^2(n-1)\sigma^2\right]} \]

Using geometric progressions 3 and 4:

\[ B_{t,t+n+1} = e^{-r_t\left\{\frac{1-(1-k)^{n+1}}{k}\right\}} \]
\[ \cdot e^{-a\left\{n-(1-k)\left[\frac{1-(1-k)^n}{k}\right]\right\}} \]
\[ \cdot e^{\frac{\sigma^2}{2}\left\{\frac{n-(1-k)^2\left[\frac{1-(1-k)^{2n}}{1-(1-k)^2}\right]}{1-(1-k)^2}\right\}} \]
Calibrating the Model to the Current Term Structure

Hull and White suggest the following generalisation:

\[ r_{t+1} - r_t = k(a_t - r_t) + \varepsilon_{t+1} \]

This is equivalent to

\[ r_{t+1} = r_t + \theta_t + k(a - r_t) + \varepsilon_{t+1} \]

where \( \theta_t \) is a time dependent drift.

With this adjustment, the model can be fitted exactly to the current term structure of bond forward prices.
A Two-factor Extension

• 1-factor model does not reflect term-structure of volatility

• 2-factor model has a stochastic central tendency

• Let

\[ r_{t+1} - r_t = k_1(a_t - r_t) + y_t + \varepsilon_{t+1} \]
\[ y_t - y_{t-1} = -k_2 y_{t-1} + \nu_t \]

\( k_1 \) and \( k_2 \) are rates of mean reversion
\( \varepsilon_{t+1} \) and \( \nu_t \) are drawings from normal distributions \((0, \sigma_1), (0, \sigma_2)\)
A Two-factor Extension

After successive substitution:

\[ r_{t+3} = (1 - k_1)^3 r_t + k_1 [a_{t+2} + (1 - k_1) a_{t+1} \]
\[ + (1 - k_1)^2 a_t] \]
\[ + y_t [(1 - k_1)^2 + (1 - k_1)(1 - k_2) + (1 - k_2)^2] \]
\[ + (1 - k_1)^2 \varepsilon_{t+1} + (1 - k_1) \varepsilon_{t+2} + \varepsilon_{t+3} \]
\[ + (1 - k_2) \nu_{t+1} + \nu_{t+2} \]

If \( \text{var}(\varepsilon_{t+i}) = \sigma_1^2 \) and \( \text{var}(\nu_{t+i}) = \sigma_2^2 \),

\[ \text{var}(r_{t+n}) = \sigma_1^2 \left[ \frac{1 - (1 - k_1)^{2n}}{1 - (1 - k_1)^2} \right] \]
\[ + \sigma_2^2 \left[ \frac{1 - (1 - k_2)^{2(n-1)}}{1 - (1 - k_2)^2} \right] \]

Annualised st.dev.

\[ \text{std}_t(r_{t+n}) = \sqrt{\text{var}_t(r_{t+n})/n} \]
Gaussian Model: Advantages and Disadvantages

• Simple model
  – Relatively easy to program (using a binomial tree)
  – Can capture the current term structure
  – Two-factor model can reflect term structure of volatilities

• Yields analytical (zero-coupon) bond prices
  – Closed form expression for bond price
  – Bond prices are lognormal
  – Options on bonds priced with BS

• But, short rates may not normally distributed

• Two-factor model may not yield realistic swaption prices
Lecture 2: Lognormal Models

• Types of Lognormal Models
  – One-Factor BK spot-rate model
  – Two-factor PSS spot-rate model
  – Forward rate models

• One-Factor BK model
  – Short rate follows a mean reverting, lognormal process
  – Under risk-neutral measure
  – Constructed using HSS (1995) method

• Two-Factor PSS model
  – Short rate follows a mean reverting, lognormal process
  – With stochastic central tendency
  – Constructed using PSS (2003) method
Short-Rate Lognormal Model

Assume that:

\[ \ln r_{t+1} - \ln r_t = k(a_t - \ln r_t) + \varepsilon_{t+1} \]

short rate of interest, \( r_t \)

long-term mean of log of short rate, \( a_t \)

rate of mean reversion, \( k, \ 0 < k < 1 \)

\( \varepsilon_{t+1} \), drawing from a normal distribution, \( E_t(\varepsilon_{t+1}) = 0, \text{var}_t(\varepsilon_{t+1}) = \sigma^2. \)

Hence,

\[ \ln r_{t+1} = ka_t + (1 - k)\ln r_t + \varepsilon_{t+1} \quad (1) \]

Equation (1) holds for all \( t \)
Mean and Variance of $ln r_{t+n}$ ($a_t = a$)

Mean of $ln r_{t+n}$:

$$E_t(ln r_{t+n}) = a[1 - (1 - k)^n] + (1 - k)^n ln r_t$$

Variance of $ln r_{t+n}$:

$$var_t(ln r_{t+n}) = \sigma^2 \left[ \frac{1 - (1 - k)^{2n}}{1 - (1 - k)^2} \right]$$

Annualised volatility

$$std_t(ln r_{t+n}) = \sqrt{var_t(ln r_{t+n})/n}$$

Mean of $r_{t+n}$:

$$E_t(r_{t+n}) = e^{E_t(ln r_{t+n}) + \frac{var_t(ln r_{t+n})}{2}}$$
Two-Factor BK Model

\[ \ln r_{t+1} - \ln r_t = k_1(a_t - \ln r_t) + \ln y_t + \varepsilon_{t+1} \]
\[ \ln y_t - \ln y_{t-1} = -k_2 \ln y_{t-1} + \nu_t \]

• \( y_t \) is a ‘premium’ factor
• a shock to the futures rate
Mean and Volatility of \( r_{t+n} \)

Since \( r_{t+n} \) is lognormal, the mean is given by

\[
E_t(r_{t+n}) = e^{E_t(ln r_{t+n}) + \frac{var_t(ln r_{t+n})}{2}}
\]

Example: 2-Factor BK model

\( t = 0 \)

short rate, \( r_0 = 0.05 \)

long term mean, \( a = 0.06 \)

mean reversion, \( k_1 = 0.15 \)

\( \sigma_1 = 0.2 \)

mean reversion, \( k_2 = 0.2 \)

\( \sigma_2 = 0.15 \)

\( \theta_t = 0 \)

\[
E_0(r_5) = e^{-2.894 + \frac{0.168^2}{2}} = 0.0602
\]

Volatility = 18.32%
Implementing the BK and 2-Factor BK Models

• One-Factor BK spot-rate model
  – Fitting the mean of the process
    * Using futures Libor quotes
    * Iterative calibration to forward bond prices
  – Calibrating to cap volatilities
    * Generalise model using $\sigma(t)$

• Two-factor BK model
  – PSS implementation
    * Recombining tree in two dimensions
  – G2++ model, Brigo and Mercurio
HSS Implementing the BK Model

- HSS (1995) build a binomial approximation to a lognormal process
- A re-combining tree with mean reversion and time-dependent volatility
- Conditional probabilities in the tree
- Method can be applied to interest rate process
HSS Implementation of the BK Model

HSS Method

- First build a process for $x_t$, where $E_0(x_t) = 1$
- Assume mean reversion and volatility same as in the interest-rate process

\[
\ln x_{t+1} = a_{x,t} + (1 - k) \ln x_t + \epsilon_{t+1}
\]

\[
a_{x,t} = E_0[\ln x_{t+1}] - (1 - k) E_0[\ln x_t] \\
= \frac{-\text{var}_0[\ln x_{t+1}]}{2} - (1 - k) \frac{-\text{var}_0[\ln x_t]}{2}
\]

- The conditional probabilities, $q_t$ depend on $a_{x,t}$ and $k$
- Scale the process using futures rates to obtain approximation to

\[
\ln r_{t+1} = \theta_t + (1 - k) \ln r_t + \epsilon_{t+1}
\]
HSS Implementation of the BK Model: an Example

- Futures rates
  
  \[ h_{0,1} = 5.0\% \]
  \[ h_{0,2} = 5.0\% \]
  \[ h_{0,3} = 5.0\% \]
  \[ h_{0,4} = 5.0\% \]

- Volatility (constant): 10%

- Mean reversion \( k = 0.2 \)

- Cap Vols
  
  \[ t = 1 : 0.1 \]
  \[ t = 2 : 0.0906 \]
  \[ t = 3 : 0.0827 \]
  \[ t = 4 : 0.0760 \]
Implemention of the 2-Factor BK Model
The PSS (2002) method

• First build a process for $x_t$, $y_t$, where $E_0(x_t) = 1$, $E_0(y_t) = 1$

• Assume mean reversion and volatility same as in the interest-rate process

• $ln x_{t+1} = a_{x,t} + (1 - k_1)ln x_t + ln y_t + \epsilon_{t+1}$
  $ln y_{t+1} = a_{y,t} + (1 - k_2)ln y_t + \nu_{t+1}$

• The conditional probabilities, $q_{x,t}$ depend on $a_{x,t}$, $k_1$, and $ln y_t$

• Scale the process using futures rates to obtain approximation to
  $ln r_{t+1} = \theta_t + (1 - k_1)ln r_t + \epsilon_{t+1}$
Implemention of the 2-Factor BK Model

The G2++ method

Brigo and Mercurio suggest a transformation of the \((r_t, y_t)\) process:

\[
\begin{align*}
    r_{t+1} - r_t &= \theta_t - k_1 r_t + y_t + \varepsilon_{t+1} \\
    y_{t+1} - y_t &= -k_2 y_t + \nu_{t+1}
\end{align*}
\]

Define

\[
\begin{align*}
    x_t &= r_t + \frac{y_t}{k_2 - k_1} \\
    \psi_t &= \frac{y_t}{k_1 - k_2}
\end{align*}
\]

Then:

\[
\begin{align*}
    x_t - x_{t-1} &= \theta_t - k_1 x_t + \eta_{t+1} \\
    \psi_t - \psi_{t-1} &= -k_2 \psi_t + \frac{\nu_{t+1}}{k_1 - k_2},
\end{align*}
\]

\[
\eta_{t+1} = \varepsilon_{t+1} + \frac{\nu_{t+1}}{k_1 - k_2}
\]
Implemention of the 2-Factor BK Model
The G2++ method

$$cov(\eta_{t+1}, \nu_{t+1}) = \frac{1}{k_2 - k_1} \text{var}(\nu_{t+1})$$

$$= \frac{1}{k_2 - k_1} \sigma_2^2$$
2-Factor BK Model
Conclusions

• BK model is too simple
• LMM is complex and lacking intuition
• 2-factor BK model is a good compromise
• Captures term structure of cap volatilities
• Generate scenarios for risk management
• Valuation of Bermudan swaptions, exotics
• 3-factor extension to capture swap-vation vols