"Cautiousness" and Measuring An Investor's Motive to Buy Options

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Abstract

In this paper we study the portfolio problem of investors who consider investments in a risk-free bond, a stock, and a stock option while maximizing the expected utility of their terminal wealth. We show that if investor i is strongly more *cautious* than j, then investor j buys an option only if i does so, and investor i sells the option only if j does so, regardless of their initial wealth, the underlying stock price, and the option price; and the reverse is also true. This shows that *cautiousness*, which is equivalent to the ratio of prudence to risk aversion, is a measure of an investor's motive to buy options. We also discuss some properties and applications of this measure. The model in this paper does not assume all investors are rational utility-maximizers.

Keywords: Cautiousness, demand for options, prudence, risk aversion.

JEL codes: D81, G1.

Introduction

Investors pursue three important financial activities, namely making savings, buying equity, and trading derivatives. How do we measure the strength of their motive in doing these activities?

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Pratt (1964) and Arrow (1965) developed the measure of risk aversion, which is defined as the negative ratio of the second derivative to the first derivative of a utility function. Pratt (1964) showed that the higher an investor's measure of risk aversion, the more risk premium he demands and the less investment he makes in the equity market.

Kimball (1990) investigated how to measure the strength of an investor's motive to make precautionary savings. He developed the measure of prudence, which is defined as the negative ratio of the third derivative to the second derivative of a utility function. The higher an investor's measure of prudence, the more precautionary saving he will make when he responds to a risk to his wealth.

An investor's motive to trade derivatives is more complicated to measure. When the measures of risk aversion and prudence are developed an investor's activities of making precautionary savings and buying stocks are separated; however, when deciding an investor's optimal position in a derivative market it makes little sense if his position in the underlying equity market is ignored. Hence we can not separate an investor's activity in the derivative market from that in the underlying equity market. This is the reason that makes it difficult to develop a measure of an investor's motive to trade derivatives.

In this paper we take on this difficult task and will show that *cautiousness*, which is equal to the ratio of prudence to risk aversion minus one, measures an investor's motive to buy options.[1] To show this, we will study the portfolio problem of investors who consider investments in a risk-free bond, a stock, and a stock option while maximizing the expected utility of their terminal wealth. We show that if investor i is strongly more *cautious* than j, that is, the lower bound of investor i's *cautiousness* is above the upper bound of j's *cautiousness*, then investor j buys an option only if i does so, and investor i sells the option only if j does so, regardless of their initial wealth, the underlying stock price, and the option price; and the reverse is also true.

In the literature on demand for options the important work by Leland (1980) is among the most frequently cited. Leland used the first derivative of risk tolerance, which is equivalent to *cautiousness*, to explain the demand for portfolio insurance. His model is based on the argument that global convexity/concavity of an investor's risk-sharing rule implies that the investor buys/sells all options. Under this argument, we need a complete set of options on the aggregate wealth with strike prices from zero to infinity, and an option buyer will buy all options with strike prices from zero to infinity while an option seller will sell all options, which may not describe option buyers and sellers very well. Thus these results are more useful to explain the convexity of risk-sharing rules in a complete market. Moreover, in Leland's model the convexity of an investor's risk-sharing rule has to be determined by comparing his preference with that of an aggregate investor. However, the aggregate investor's preference is, in general, unknown and can only be assumed exogenously (unless all investors are rational expected utility maximizers and their preferences and beliefs are all known).

[1] For the definition of cautiousness, see Wilson (1968).

The model presented in this paper directly studies the portfolio problem of investors who consider investments in a risk-free bond, a stock, and a stock option. This overcomes the above problems which are common to the models which explain demand for options by studying the convexity of risk-sharing rules. We do not need to know the preference of a representative investor, neither do we need to require that all investors be utility-maximizers. We do not even assume that all investors are rational. All we assume is that there are some investors who are rational expected-utility-maximizers, and whose behaviors in the options market are the subject of the research in this paper. Moreover, we do not require market completeness. We assume there is one option which the investigated investor is considering in his optimal investment strategy. This way of modeling is in line with the approach we use when we study the demand for equity. For example, when Pratt (1964) and Arrow (1965) developed the measure of risk aversion they assumed there is one risky asset and then showed how risk aversion affects an investor's demand for this asset.

Apart from Leland (1980), there are some papers in the literature which are closely related to this work. For example, Gollier (2001) discussed how investors' cautiousness measures are related to the local convexity of their consumption rules. He showed that an agent's consumption rule is locally convex (concave) if his cautiousness is larger (smaller) than the weighted average cautiousness measure.[2] In an earlier related study, Carroll and Kimball (1996) investigated the effect of uncertainty on the curvature of investors' consumption rules by examining the effect of uncertainty on their cautiousness measures. They showed that if investors have HARA class utility functions, then uncertainty will increase their cautiousness measures, which leads to concavifition of their optimal consumption rules.

Another important study which we refer to is the paper written by Franke, Stapleton, and Subrahmanyam (hereafter FSS) (1998). They used background risk to explain the demand for options. They applied the same argument as Leland (1980), i.e., global convexity/concavity of an investor's risk-sharing rule implies that the investor buys/sells all options. The economy they studied is a special one in which investors have identical positive constant *cautiousness*. They showed that in such an economy the investors without background risk have globally concave risk-sharing rules; however, it is not necessary that the investors with background risk have globally convex risk-sharing rules.[3]

In this paper as an application of our main result, we also discuss the impact of background risk on the demand for options. We show that if an investor has HARA class utility then when he has additive (multiplicative) background risk, the *cautiousness* of his derived utility function will be strictly higher unless he has exponential (power) utility, hence he will have a stronger motive to buy options. Unlike the model given by FSS (1998), the above result in this paper is valid when there is no complete market of contingent claims on the stock and even valid when many other investors are not rational utility maximizers. This

[2]See Gollier (2001) page 207, Proposition 52.

[3]Unlike in Leland's model, the preference of the representative investor is derived endogenously. See also Appendix A of this paper for the discussion of their framework.

result can also be used to extend the result given by FSS (1998) to the case where some investors have additive or/and multiplicative background risk. The extension is presented in Appendix A.

In another interesting application of our main result, we use cautiousness to explain the demand for stocks and corporate bonds. We show that if investor i is strongly more cautious than investor j then investor j will not buy more shares than corporate bonds unless investor i does so and investor i will not buy less shares than corporate bonds unless investor j does so regardless of the investors' initial wealth, the initial stock price and corporate bond price; and the reverse is also true.

More recently, Hara, Huang, and Kuzmics (hereafter HHK) (2007) further investigated the relationship between the curvature of an investor's risk -sharing rule and his cautiousness. By comparing individual investors' preferences with that of the representative investor which is derived endogenously, they showed that although an investor with higher cautiousness measure may be more likely to have convex risk-sharing rules, in general investors rarely have globally convex or concave risk-sharing rules except for those who have the highest or lowest cautiousness measures.[4]

Other related studies include Brennan and Solanki (1981), Benninga and Blume (1985), Brennan and Cao (1996), and Carr and Madan (2001). Similar to Leland (1980), Brennan and Solanki also investigated the demand for portfolio insurance, however they focused on the the case where state prices are derived from risk neutral valuation relationships and the portfolio returns are lognormally distributed. Benninga and Blume investigated the optimality of a certain insurance strategy in which an investor buys a risky asset and a put on that asset. Brennan and Cao investigated the impact of asymmetric information on the demand for options in an economy with exponential utility and normally distributed returns. They concluded that well informed investors tend to buy options on good news and sell options on bad news. Carr and Madan discussed how investors' preferences and beliefs affect their positions in derivative markets.

The structure of this paper is as follows. In the first section we introduce the concept of *cautiousness* and the definition of being strongly more cautious. In the second section we present the model. In Section three we present our main result that *cautiousness* is a measure of an investor's motive to buy options. In the fourth section we give some numerical examples to illustrate the main result. In Section five we discuss some properties of *cautiousness*. In the sixth section we present two applications of the main result: (1) how does background risk affect an investor's motive to buy options? (2) who buys more stocks and who buys more bonds? The final section concludes the paper.

[4]This further highlights the difficulty in using the convexity of risk-sharing rules to explain the demand for options.

1 Cautiousness

To introduce the concept of *cautiousness* we first have to explain the concepts of risk aversion and prudence. Pratt (1964) and Arrow (1965) developed the concept of risk aversion to explain investors' behavior in the equity market. As interpreted by Pratt (1964), given utility function $u(x)$, the function $R(x) = -u''(x)/u'(x)$ is a measure of risk aversion.[5] The higher risk aversion an investor has, the larger risk premium he demands for a small and actuarially neutral risk. More precisely, the risk premium demanded by an investor with utility $u(x)$ will be approximately the function $R(x)$ times half the variance of the risk. It is also shown to be a global measure of risk aversion in the sense that if the function $R(x)$ of an investor is always larger than that of the other, then the former will demand a larger risk premium than the latter for any risk, large or small, at any wealth level.

Kimball (1990) developed a theory about investors' precautionary savings analogous to Pratt's (1964) theory of risk aversion. Absolute prudence is defined as $P(x) = -u'''(x)/u''(x)$. The higher prudence an investor has, the more equivalent precautionary premium he demands for a risk to his wealth and the more precautionary savings he makes in response to the risk.

The first derivative of risk tolerance, where risk tolerance is the inverse of absolute risk aversion, was called *cautiousness* by Wilson (1968)[6]. Given a utility function, $u(x)$, its *cautiousness* is $C(x) \equiv (1/R(x))' = (-u'(x)/u''(x))'$. Equivalently it can be defined as the ratio of absolute prudence to absolute risk aversion minus one. This can be shown as follows.

Given an increasing and concave utility function $u(x)$, we have

$$(\ln R(x))' = (\ln -u''(x))' - (\ln u'(x))' = -(P(x) - R(x))$$

which can be written as

$$R'(x) = -R(x)(P(x) - R(x)).$$

It follows that

$$(1/R(x))' = -R'(x)/R^2(x) = P(x)/R(x) - 1.$$

Thus we have $C(x) = P(x)/R(x) - 1$.

More explicitly we can write it as

$$C(x) = u'''(x)u'(x)/u''^2(x) - 1.$$

Note that

$$(R(x))' = -R^2(x)(1/R(x))' = -R^2(x)C(x).$$

[5] Throughout this paper we assume all utility functions are strictly increasing, strictly concave, and three times differentiable.

[6] See page 129, Wilson (1968).

Thus decreasing absolute risk aversion (hereafter DARA) is equivalent to positive *cautiousness* and constant absolute risk aversion (hereafter CARA) is equivalent to zero *cautiousness*.

It is well known that exponential utility functions have zero *cautiousness* while other HARA utility functions have constant positive *cautiousness*. For example, given a HARA utility function $u(x) = (x + a)^{1-\gamma}/(1 - \gamma)$, where $\gamma > 0$, we have constant cautiousness $C(x) = 1/\gamma$, regardless of the substance parameter a. Note the risk aversion measure of the above HARA utility is $R(x) = \gamma/(x + a)$. Thus if we change the value of a, the risk aversion measure will vary while the cautiousness measure remains constant. From this example, it is clear that a utility function has a higher cautiouness measure does not necessarily have a lower risk aversion measure.

Now we define a key concept in this paper.

Definition 1 *Investor i is said to be strongly more cautious than investor j if there exists a constant C such that for all w and v, $C_i(w) \geq C \geq C_j(v)$, where $C_i(w)$ and $C_j(v)$ are the coefficients of cautiousness of investors i and j respectively.*[7]

It is straightforward that the condition in the definition is equivalent to $\inf_w C_i(w) \geq \sup_v C_j(v)$. The above concept gives an ordering of utility functions in terms of their *cautiousness*. Since HARA class utility functions have constant *cautiousness* thus they can be ordered perfectly in this way.

2 The Model

Assume a two-date economy with starting time 0 and ending time 1. Assume there is a risk-free bond traded in the market; the risk-free interest rate is denoted by r. Assume there is a stock available in the market whose prices at time 0 and 1 are denoted by S_0 and S respectively.

Assumption 1 *Assume the distribution of the stock price S is continuous and its support is an interval in $[0, +\infty)$.*

We denote the support interval by I. The interval I can be either bounded or unbounded. Although we assume that the stock price follows a continuous distribution, the result can be easily extended to the discrete case.

Assume there is a convex derivative written on the stock that is traded in the market. Here we first clarify the concept of convex derivative.

Definition 2 *A derivative written on a stock is called to be convex if its payoff function is piecewise continuously differentiable and everywhere convex in the stock price $S \in I$, and the convexity is strict for at least one point.*

[7]Through out the paper, when we say for all w and v, $C_i(w) \geq C \geq C_j(v)$, we mean for all w and v in the natural domains of $u_i(x)$ and $u_j(x)$ respectively.

The above definition ensures that the convex derivative will not degenerate to a fraction of the stock; thus it ensures that the investment problem of allocating money to the bond, the stock, and the convex derivative will not degenerate to the problem of allocating money to the bond and the stock only.

Note for a call or a put option with strike price K inside the interval I, its payoff function is convex and the convexity is strict at K. According to the above definition, such an option is a convex derivative. However, if K is not inside I, then the option degenerates to the underlying stock, thus it is not a convex derivative.

Denote the payoff of the derivative at time 1 by $c(S)$. Note since $c(S)$ is a convex function of S, $c(S)$ is continuous. Denote the price of the derivative at time 0 by c_0. The interest rate and the current prices of the stock and the derivative are determined in the equilibrium of the market. For an individual investor, he can only take them as given from the market.

We stress here that we do not assume all investors are rational utility-maximizers. We only assume there are some investors who are rational expected-utility-maximizers whose behaviors in the option market are the subject of this research. After all, an expected-utility framework can only deal with those who are utility maximizers. These investors are indexed by $i = 1, 2, \ldots$; and they are all price-takers. Investor i's preference is represented by utility function $u_i(x)$. At time 0 he has initial wealth w_{0i}. Assume investor i buys x_i shares of the stock and y_i units of derivatives, and invests the rest of his money in the money market, which is $w_{0i} - x_i S_0 - y_i c_0$. Denote investor i's wealth at time 1 by $w_i(S; x_i, y_i)$. We have

$$w_i(S; x_i, y_i) = (w_{0i} - x_i S_0 - y_i c_0)(1 + r) + x_i S + y_i c(S).$$

For brevity we often write $w_i(S; x_i, y_i)$ simply as $w_i(S)$.

Investor i maximizes the expected utility of his ending time wealth $w_i(S)$ while requiring the ending-time wealth being non-negative. That is,

$$\max_{x_i, y_i} Eu_i(w_i(S)), \quad s.t.\ w_i(S) \geq 0. \tag{1}$$

We obtain the first order conditions for an interior solution:

$$Eu_i'(w_i(S))(S - (1 + r)S_0) = 0, \quad \text{and} \quad Eu_i'(w_i(S))(c(S) - (1 + r)c_0) = 0,$$

which can be written as

$$\frac{E[u_i'(w_i(S))S]}{Eu_i'(w_i(S))} = (1 + r)S_0, \quad \text{and} \quad \frac{E[u_i'(w_i(S))c(S)]}{Eu_i'(w_i(S))} = (1 + r)c_0. \tag{2}$$

The solution, (x_i, y_i), depends on the utility function, (S_0, c_0), and the initial wealth of the investor given the interest rate r and the distribution of the stock price. We now make the following assumptions.

Assumption 2 *All utility functions are strictly increasing, strictly concave, and three times differentiable.*

The strict concavity of the utility functions guarantees that the second order condition for the expected utility maximization problem is always satisfied and a solution to (2) is a global maximum which is unique. This is summarized in the following Lemma.

Lemma 1 *A solution to (2) is a solution to (1). The solution is unique.*

Proof: We need only show that the objective function of the maximization problem (1) is a strictly concave function of all x_i and y_i. To show this we need to calculate the Hessian matrix H. Let $f(x_i, y_i)$ denote the objective function of (1). We have

$$H = \left\{ \begin{array}{cc} \frac{\partial^2 f}{\partial x_i^2} & \frac{\partial^2 f}{\partial x_i \partial y_i} \\ \frac{\partial^2 f}{\partial x_i \partial y_i} & \frac{\partial^2 f}{\partial y_i^2} \end{array} \right\},$$

where $\frac{\partial^2 f}{\partial x_i^2} = Eu_i''(w_i(S))(S - S_0(1+r))^2$, $\frac{\partial^2 f}{\partial x_i \partial y_i} = Eu_i''(w_i(S))(S - S_0(1+r))(c(S) - c_0(1+r))$, and $\frac{\partial^2 f}{\partial y_i^2} = Eu_i''(w_i(S))(c(S) - c_0(1+r))^2$. According to the Cauchy-Schwarz inequality, the determinant $|H| \geq 0$, where equality holds if and only if $(S - S_0(1+r))$ and $(c(S) - c_0(1+r))$ are linearly dependent. But as $c(S)$ is strictly convex for at least one point, we must have $|H| > 0$. As we also have $\frac{\partial^2 f}{\partial x_i^2} < 0$, H must be negative definite everywhere, which implies that $f(x_i, y_i)$ is strictly concave everywhere. Q.E.D.

Before we proceed to the next section, we first introduce some notation. Let $R_i(x)$ denote investor i's absolute risk aversion, i.e., $R_i(x) \equiv -u_i''(x)/u_i'(x)$. Let $C_i(w)$ denote his coefficient of *cautiousness*, i.e., $C_i(w) \equiv (1/R_i(x))'$.

Let $\phi_i(S) \equiv u_i'(w_i(S))/Eu_i'(w_i(S))$. Then (2) can be written as

$$E[\phi_i(w_i(S))S] = (1+r)S_0, \quad \text{and} \quad E[\phi_i(w_i(S))c(S)] = (1+r)c_0. \tag{3}$$

Thus $\phi_i(S)$ can be regarded as investor i's pricing kernel, which he uses to price the stock and the derivative. As the investor has to take the market prices as given his individual pricing kernel must price the stock and the derivative correctly; that is, an individual pricing kernel must be admissible. As is well known, this individual pricing kernel is the Radon-Nikodym derivative of an equivalent Martingale measure with respect to the true probability measure. Thus there must exist an equivalent Martingale measure $Q_i(S)$ such that $\phi_i(S) = \frac{dQ(S)}{dP(S)}$, where $P(S)$ denote the true probability measure.

Note if the market is complete, then there is a unique equivalent Martingale measure, which leads to a unique admissible pricing kernel — the market pricing kernel (or the representative investor's pricing kernel); thus all individual pricing kernels must be equal to this pricing kernel. In this case if we are given the market pricing kernel, as in Leland (1980) and Brennen and Solanki (1981), there will be a one-to-one relationship between an investor's risk sharing rule and his utility function. Thus it will be straightforward to obtain the relation between an investor's demand for the derivative and his utility function, as Leland (1980) and Brennen and Solanki (1981) did. However, when the market is incomplete, admissible pricing kernels are not unique. An individual pricing kernel can be any one of the many admissible pricing kernels. There will not be a one-to-one relationship between an investor's risk sharing rule and his utility function. Thus it is no longer so easy to obtain the relation between an investor's demand for the derivative and his risk preferences.

Since the payoff of the derivative, $c(S)$, is piecewise continuously differentiable, so is $w_i(S)$.

Let $\delta_i(S) \equiv -\phi_i'(S)/\phi_i(S)$. We have

$$\delta_i(S) = R_i(w_i(S))w_i'(S). \tag{4}$$

3 Measuring the Motive to Buy Options

In this section we will show that *cautiousness* is a measure of an investor's motive to buy options. We now present our first main result.

Theorem 1 (Sufficiency) *Assume there is an interior solution to the investment problem (1) for both investors i and j. If investor i is strongly more cautious than investor j, then investor j buys the derivative only if investor i does so, and investor i sells the derivative only if investor j does so, regardless of their initial wealth, the stock price, and the derivative price.*

Before we proceed to prove this result, we first explain the significance of the statements in the theorem. The above result essentially states that if investor i is strongly more *cautious* than investor j, then investor i always has a stronger motive to buy the derivative regardless of their initial wealth, the stock price, and the derivative price.

Note an investor with higher coefficient of *cautiousness* does not necessarily have higher risk aversion ratio. In fact, given a certain level of *cautiousness*, we can make an investor arbitrarily less risk-averse. For example, suppose an investor has HARA class utility function $u(x) = \frac{(x+a)^{1-\gamma}}{1-\gamma}$, where $\gamma > 0$ and a are constant. Obviously the investor has constant *cautiousness* $1/\gamma$ for any a; however, the investor's risk aversion ratio is equal to $\frac{\gamma}{x+a}$, which depends on a. It is straightforward to see that increasing a to infinity we can make the investor extremely less risk averse; the conclusion from the theorem is that however less risk averse the investor may become, it can never change his position in the demand and supply chain of the derivative. This is really a big surprise considering that many people think that risk aversion explains an investor's decision to buy or sell options.

Another observation is that for an exponential utility function, its cautiousness is zero while any utility function which is DARA has positive cautiousness. Thus according to the above theorem, any investor who has decreasing absolute risk aversion always has stronger motive to buy options than an investor with an exponential utility function.

The proof of the above theorem is difficult; however, the idea is clear. Note if an investor buys (sells) the option, then his payoff function will be convex (concave). Consider investor i who is strongly more cautious than investor j. We need only to show that his payoff function can not be concave if investor j's payoff function is convex. The key is to prove the following statement:

"if investor i is strongly more cautious than investor j and sells the derivative while investor j does not then investor i's individual pricing kernel will be more convex than investor j's (strictly more convex for at least one point)."

But as is well known, if one pricing kernel is more convex than another, then the former will give higher option prices than the later, i.e., the two pricing kernels can not both price the derivative correctly, which leads to contradiction.

If the payoff function of the derivative is twice differentiable in the entire support, the proof of the above statement will be much easier. Recall from (4) we have

$$\delta_i(S) = R_i(w_i(S))w_i'(S).$$

If $w_i(S)$ were twice differentiable, differentiating the above equation, we would obtain

$$\delta_i'(S) = -C_i(w_i(S))\delta_i^2(S) + R_i(w_i(S))w_i''(S).$$

Rewrite it as

$$(\frac{1}{\delta_i(S)})' = C_i(w_i(S)) - R_i(w_i(S))w_i''(S)/\delta_i^2(S).$$

Thus if investor i is strongly more cautious than investor j and sells the derivative while investor j does not, then $C_i(w_i(S)) \geq C_j(w_j(S))$, $w_i''(S) \leq 0 \leq w_j''(S)$ and the latter inequality is strict for at least one point (S^*). It follows that $(\frac{1}{\delta_i(S)})' \geq (\frac{1}{\delta_j(S)})'$ and the inequality is strict for at least one point (S^*). That is, investor i's individual pricing kernel is more convex than investor j's and strictly more convex at S^*. Then as the derivative's payoff is conex and strictly convex at S^*, it is well known that the former pricing kernel prices the derivative strictly higher than the latter. This causes contradiction as both individual pricing kernels are admissible.

When the payoff function of the derivative is not twice differentiable in the entire support but is piecewise twice differentiable, the proof of the above statement is similar in spirit but has much more technicality, as is shown below.

Proof:

Note $c(S)$ is globally convex in S and there exists at least one point, S^*, at which $c(S)$ is strictly convex. Thus investor j buys (sells) the derivative if and only if his optimal strategy is convex (concave) and the convexity (concavity) is strict for at least one point, S^*.

Suppose investor i sells the derivative but investor j does not do so, then $w_i(S)$ is concave, and for at least one point, S^*, the concavity is strict; while $w_j(S)$ is convex.

Recall from (4) we have

$$\delta_i(S) = R_i(w_i(S))w_i'(S).$$

If $w_i(S)$ were twice differentiable, differentiating the above equation, we would obtain

$$\delta_i'(S) = -C_i(w_i(S))\delta_i^2(S) + R_i(w_i(S))w_i''(S).$$

However, since $w_i(S)$ is not necessarily twice differentiable, we do not have the above result. Nevertheless, when S increases, the increase of $\delta_i(S)$ still consists of two parts: one is from the increase of $R_i(w_i(S))$, and the other is from the increase of $w_i'(S)$. Let ΔS denote a small positive change in S. Then $\delta_i(S+\Delta S) - \delta_i(S)$ is equal to

$$\begin{aligned}&R_i(w_i(S+\Delta S))w_i'(S+\Delta S) - R_i(w_i(S))w_i'(S)\\&= (R_i(w_i(S+\Delta S)) - R_i(w_i(S)))w_i'(S) + R_i(w_i(S+\Delta S))(w_i'(S+\Delta S) - w_i'(S)).\end{aligned}$$

As $\delta_i(S+n\Delta S) - \delta_i(S)$ is equal to

$$\sum_{k=1}^{n}[R_i(w_i(S+k\Delta S))w_i'(S+k\Delta S) - R_i(w_i(S+(k-1)\Delta S))w_i'(S+(k-1)\Delta S)],$$

from the preceding equation we can write $\delta_i(S+n\Delta S) - \delta_i(S)$ as

$$\begin{aligned}&\sum_{k=1}^{n}(R_i(w_i(S+k\Delta S)) - R_i(w_i(S+(k-1)\Delta S)))w_i'(S+(k-1)\Delta S)\\&+\sum_{k=1}^{n}R_i(w_i(S+k\Delta S))(w_i'(S+k\Delta S) - w_i'(S+(k-1)\Delta S)).\end{aligned}$$

The first sum can be rewritten as

$$\sum_{k=1}^{n}\frac{R_i(w_i(S+k\Delta S)) - R_i(w_i(S+(k-1)\Delta S))}{\Delta S}w_i'(S+(k-1)\Delta S)\Delta S.$$

Let $\Delta S = \frac{\tau}{n} > 0$, where $\tau > 0$ is a constant. Let $n \to \infty$; then the first sum is equal to

$$-\int_S^{S+\tau}\frac{dR_i(w_i(S))}{dS}w_i'(S)dS.$$

As $\frac{dR_i(w_i(S))}{dS} = -R_i^2(w_i(S))C_i(w_i(S))w_i'(S)$, the above integral can be rewritten as

$$-\int_S^{S+\tau}R_i^2(w_i(S))C_i(w_i(S))w_i'^2(S)dS.$$

Hence the first sume is equal to

$$-\int_S^{S+\tau}C_i(w_i(s))\delta_i^2(s)ds, \tag{5}$$

where $\delta_i(S) = R_i(w_i(S))w_i'(S)$.

Moreover, as $w_i(S)$ is concave, every term in the second sum is non-positive; thus the second sum is no larger than

$$\pi_i(S+\tau, S) = \inf_{0\le x\le\tau} R_i(w_i(S+x))(w_i'(S+\tau) - w_i'(S)). \tag{6}$$

From (5) and (6) we obtain

$$\delta_i(S+\tau) - \delta_i(S) \le -\int_S^{S+\tau} C_i(w_i(s))\delta_i^2(s)ds + \pi_i(S+\tau, S), \tag{7}$$

where $\pi_i(S+\tau, S)$ is defined in (6), which is always non-positive and is strictly negative for $S+\tau > S^* > S$ since $w_i(S)$ is concave in the whole support and is strictly concave at S^*.

Similarly we have

$$\delta_j(S+\tau) - \delta_j(S) \ge -\int_S^{S+\tau} C_j(w_j(s))\delta_j^2(s)ds + \pi_j(S+\tau, S), \tag{8}$$

where

$$\pi_j(S+\tau, S) = \inf_{0\le x\le\tau} R_j(w_j(S+x))(w_j'(S+\tau) - w_j'(S)),$$

which is always non-negative since $w_i(S)$ is convex.

First consider any interval I in which the payoff of the derivative $c(S)$ is continuously differentiable. Suppose at one point in this interval, say S, we have $\delta_i(S) = \delta_j(S)$. If S increases slightly by a small τ, since $C_i(w_i(s)) \ge C_j(w_j(s))$ and $\pi_i(S+\tau, S) \le \pi_j(S+\tau, S)$ then from (7) and (8), $\delta_i(S)$ decreases faster than $\delta_j(S)$, and we will have $\delta_i(S+\tau) \le \delta_j(S+\tau)$. We assert that the above inequality is true not only for small $\tau > 0$ but also for all $\tau \in \{\tau | \tau > 0,\ S+\tau \in I\}$. This is because after $\delta_i(S)$ becomes smaller than $\delta_j(S)$, if it somehow increases to the point such that they are close to each other again, then again $\delta_i(S)$ decreases faster than $\delta_j(S)$, and $\delta_i(S)$ stays smaller than $\delta_j(S)$ in the whole interval.

Now consider at the points where $c'(S)$ has jumps. These jumps will cause the jumps in $\delta_i(S)$ and $\delta_j(S)$ simultaneously. Since $\delta_i(S) = R_i(w_i(S))w_i'(S)$, where $R_i(w_i(S))$ is positive and globally continuous while $w_i'(S)$ is decreasing, when $\delta_i(S)$ jumps, it jumps down. For the opposite reason, when $\delta_j(S)$ jumps, it jumps up.

Hence combining the above two cases, we conclude that $\delta_i(S) - \delta_j(S)$ changes its sign at most once from positive to negative. Because of the strict concavity of $w_i(S)$ at $S = S^*$ we conclude that there exists a neighborhood of S^*, A, such that $\delta_i(S) - \delta_j(S) \ne 0$ for all $S \in A - \{S^*\}$.

It follows that $\phi_i(S) - \phi_j(S)$ can change its sign at most twice. But since these two pricing kernels both price the stock correctly, $\phi_i(S) - \phi_j(S)$ must change its sign at least twice in the whole support of the distribution of S. Thus it changes its sign exactly twice. This implies that there exist S_1 and S_2, where $S_1 < S_2$, such that $\phi_i(S) - \phi_j(S) \ge 0$, when $S < S_1$, or $S_2 < S$; $\phi_i(S) - \phi_j(S) \le 0$, when $S_1 < S < S_2$. Moreover, there must exist a neighborhood of S^*, A_1, such that $\phi_i(S) - \phi_j(S) \ne 0$ for all $S \in A_1 - \{S^*\}$.

Now construct a portfolio of the money instrument and the stock such that its payoff equal to the payoff of the convex derivative at S_1 and S_2. Denote the payoff of the portfolio by $L(S)$. It follows from the strict concavity of $w_i(S)$ at $S = S^*$ that there exists a neighborhood of S^*, A_2, such that $c(S) - L(S) \neq 0$ for all $S \in A_2 - \{S^*\}$.

It can be verified that $\phi_i(S) - \phi_j(S)$ and $c(S) - L(S)$ always have the same sign. This, together with the conclusion that neither of them will be zero for all $S \in A_1A_2 - \{S^*\}$ and the assumption that the probability mass of $A_1A_2 - \{S^*\}$ is positive, implies that

$$E(\phi_i(S) - \phi_j(S))c(S) = E(\phi_i(S) - \phi_j(S))(c(S) - L(S)) > 0.$$

This contradicts the condition that the two pricing kernel both price the derivative correctly. Thus the initial supposition that investor i sells the derivative but investor j does not do so cannot hold. That is, if investor i sells the derivative, so must investor j. For the same reason, if investor j buys the derivative, so must investor i. This completes the proof. Q.E.D.

The theorem tells us that if investor i is strongly more *cautious* than investor j, then investor i always has a stronger motive to buy the derivative regardless of their initial wealth, the stock price, and the derivative price. Obviously it could happen that one investor is strongly more *cautious* than another investor while they both buy or sell the convex derivative. The question is, in this case could their coefficients of *cautiousness* tell something about who holds more position in the derivative? The answer is, however, more likely to be negative. We believe that *cautiousness* can not tell this. It is well known that risk aversion determines the amount of investment in risky assets; however, since *cautiousness* is not necessarily dependent on risk aversion, it will be surprising if it can tell how much position an investor will hold in the derivative.

Perhaps the above question is also related to the following question. Is *cautiousness* really a measure of cautiousness? We understand that a more cautious investor may have stronger motive to buy a protective put. That is, a more cautious investor may tend to use the convex derivative to make a portfolio insurance. Thus the question become, can we tell by an investor's coefficient of cautiousness that whether he buys the options to hedge against stock risk? The answer is, again, more likely to be negative. *Cautiousness* may tell who buys options but it may not tell what the options are bought for: they may be used to hedge the risk in the underlying stock; they may be also used for their leverage effect.

From the theorem we can immediately infer the following two corollaries.

Corollary 1 *Assume there is an interior solution to the investment problem (1) for both investors i and j. If investors i and j have the same constant coefficient of cautiousness, then they either both buy the option, or both sell the option, or both hold zero position in the option regardless of their initial wealth, the stock price, and the derivative price.*

Indeed, Rubinstein (1974) has proved a stronger result than the above corollary. He has shown that if investors i and j have the same constant coefficient of cautiousness, then they will have the same portfolio of risky assets, here being the stock and the derivative.

Corollary 2 *Assume there is an interior solution to the investment problem (1) for both investors i and j. If either of the following three conditions is satisfied: (i) investor i has decreasing absolute risk aversion while investor j has exponential utility, (ii) investor i has exponential utility while investor j has increasing absolute risk aversion, (iii) investor i has decreasing absolute risk aversion while investor j has increasing absolute risk aversion, then investor j buys the derivative only if investor i does so, and investor i sells the derivative only if investor j does so, regardless of their initial wealth, the stock price, and the derivative price.*

The above corollary is an immediate consequence of Theorem 1 as decreasing, constant, and increasing absolute risk aversion implies positive, zero, and negative cautiousness respectively.

Theorem 1 shows that being strongly more *cautious* is a sufficient condition for an investor to have stronger motive to buy options regardless of his initial wealth, the stock price, and the derivative price. We now try to show this condition is also necessary. For purely technical reasons, we now make the following two assumptions.

Assumption 3 *Assume all utility functions have continuous third derivative.*

Assumption 4 *The support of the stock price distribution, $[a, b]$, is bounded.*

Assumption (3) is to ensure that the cautiousness measure is always continuous. Assumption (4) can be easily relaxed, and the case with unbounded support of the distribution can be dealt with by making minor adjustment.

We recall that we have assumed that the expected-utility-maximizing investors in the market are strictly risk-averse. In a rare case, when the current prices of the stock and the derivative are equal to the risk neutral prices, a strictly risk averse investor will optimally hold zero investment in both the stock and the derivative. If we use S_r and c_r to denote the risk neutral prices of the stock and the derivative respectively, when $(S_0, c_0) = (S_r, c_r)$, an interior solution to (1) is $(x_i, y_i) = (0, 0)$. We now show that for those (S_0, c_0) which are near (S_r, c_r), interior solutions to (1) exist too.

Lemma 2 *There exists a neighborhood of (S_r, c_r), A, such that for any $(S_0, c_0) \in A$, an interior solution to (1) exists.*

Proof: We need only prove that there exists a neighborhood of (S_r, c_r), A, such that for any $(S_0, c_0) \in A$, a solution to (3) exists.

Since the support of the stock price distribution is bounded, the price of the stock and the derivative under the first stochastic dominance rule is bounded. Let $\underline{S}$ and $\bar{S}$ be the lower and upper bounds of the stock price; let $\underline{c}$ and $\bar{c}$

be the lower and upper bounds of the derivative price.[8] We now define a a function $f(.)$ on $[0,+\infty)\times[0,+\infty)$ as follows. For a pair of stock price and derivative price (S_0, c_0), if there is a solution (x_i, y_i) to (3), then $f(S_0, c_0) = (x_i, y_i)$. Note according to Lemma 1, the solution (x_i, y_i) is unique; thus the function is well defined. As utility functions are three times differentiable, $f(.)$ is obviously sequentially continuous. But in a metric space sequential continuity and continuity are equivalent; thus $f(.)$ is continuous.

Consider the opposite problem in which given a pair of (x_i, y_i), we want to solve (3) for (S_0, c_0). We assert that there exists a neighborhood of $(0,0)$, B, such that for any $(x_i, y_i)\in B$, the solution of (S_0, c_0) exists. This can be shown as follows. Consider the function

$$g(S_0, c_0) = \frac{1}{1+r}(E[\phi_i(w_i(S))S], E[\phi(w_i(S))c(S)]).$$

For any pair of (x_i, y_i) which is close to $(0,0)$ enough, the function is well defined on $[\underline{S}, \bar{S}]\times[\underline{c}, \bar{c}]$. As utility functions are three times differentiable, $g(.)$ is obviously a sequentially continuous function, thus a continuous function of a non-empty, closed, bounded, convex set $[\underline{S}, \bar{S}]\times[\underline{c}, \bar{c}]$ into itself. According to well-known Brouwer's Fixed Point Theorem, there is always a fixed point. Thus a solution to (3) always exists. This proves the assertion.

Hence we conclude that there is a neighborhood of $(0,0)$, B, such that B is a set of images under function $f(.)$. Since $f(.)$ is continuous and B is open, the preimage of B is also open. Thus there is a neighborhood of (S_r, c_r), A, such that for any $(S_0, c_0)\in A$, a solution to (3) exists. Q.E.D.

Theorem 2 (Necessity) *Assume investor j buys the derivative only if investor i does so, and investor i sells the derivative only if investor j does so, regardless of their initial wealth and current prices of the stock and the derivative. Then investor i is strongly more cautious than investor j.*

If there does not exist a constant C such that for all w and v, $C_i(w)\geq C\geq C_j(v)$, then there must exist some w_0 and v_0 such that $C_i(w_0) < C_j(v_0)$. As according to Assumption (3), cautiousness is continuous, there must be a neighborhood of w_0, A, a neighbor hood of v_0, B, and a constant α, such that for all $w\in A$ and all $v\in B$, $C_i(w) < \alpha < C_j(v)$. If we can somehow make sure that investor i's terminal wealth is contained in A while investor j's terminal wealth is contained in B, then using an argument very similar to the proof of Theroem 1, we can show a situation where it happens that investor j optimally holds a long position in the derivative while i does not. This is the idea we use to prove the above theorem.

[8] It is straightforward that $\underline{S} = \frac{a}{1+r}$ and $\overline{S} = \frac{b}{1+r}$. It can also be proved that the lower bound on the derivative price is attained under a single point distribution with support $\{S_0(1+r)\}$ while the upper bound on the derivative price is attained under a two-point distribution with support $\{a, b\}$. Thus it can be calculated that $\underline{c} = \max\{0, \frac{c(S_0(1+r))}{1+r}\}$ and $\overline{c} = \frac{p_a c(a)}{1+r} + \frac{(1-p_a)c(b)}{1+r}$, where $p_a = \frac{b-(1+r)S_0}{b-a}$. The proof of this result is omitted, but is available on request.

Proof:

We need only to show that if there does not exist a constant C such that for all w and v, $C_i(w) \geq C \geq C_j(v)$ then there is a set of w_{i0}, w_{j0}, S_0, and c_0 such that investor j optimally holds a long position in the derivative while i does not.

When $y_i = 0$, the first equation in (2) becomes

$$\frac{1}{1+r}\frac{E[u_i'(w_i(S))S]}{Eu_i'(w_i(S))} = S_0$$

where $w_i(S; x_j, 0) = (w_{0i} - x_i S_0)(1+r) + x_i S$.

As in the proof for Lemma 2, we can easily show that for any small $x_n > 0$, a solution of S_0 to the above equation exists. This implies that there is a series: $\{(x_i^n, 0) | n = 1, 2, ...\}$, where x_i^n is strictly decreasing in n, $\lim_{n\to\infty} x_i^n = 0$, and for all n, $(x_i^n, 0)$ is the solution to (2) corresponding to $(S_0, c_0) = (S_{0n}, c_{0n})$. Obviously we have

$$\lim_{n\to\infty} S_{0n} = S_r \quad \text{and} \quad \lim_{n\to\infty} c_{0n} = c_r.$$

When n is sufficiently large, x_i^n is sufficiently small; these solutions are obviously interior solutions to (1). Without loss of generality assume for all n, $(x_i^n, 0), n = 1, 2, ...$, are interior solutions.

According to Lemma 2, there exists a neighborhood of (S_r, c_r), A, such that for any $(S_0, c_0) \in A$, the solution to (2) exists. Without loss of generality assume for all n, $(S_{0n}, c_{0n}) \in A$.

Applying Lemma 2 we conclude that given the series $\{(S_{0n}, c_{0n}) | n = 1, 2, ...\}$, there also exist a series of interior solutions $\{(x_{jn}, y_{jn}) | n = 1, 2, ...\}$ to (1) for investor j. Since $\lim_{n\to\infty}(S_{0n}, c_{0n}) = (S_r, c_r)$ from the continuity of the solutions we have

$$\lim_{n\to\infty}(x_{jn}, y_{jn}) = (0, 0).$$

Let investors' optimal strategies and pricing kernels corresponding to (S_{0n}, c_{0n}) be marked by an additional subscript n.

As is pointed out in the paragraph preceding this proof, since there does not exist a constant C such that for all w and v, $C_i(w) \geq C \geq C_j(v)$, there must be w_0, v_0, a neighborhood of w_0, A, a neighbor hood of v_0, B, and a constant α, such that for all $w \in A$ and all $v \in B$, $C_i(w) < \alpha < C_j(v)$. Let $w_{i0} = w_0/(1+r)$ and $w_{j0} = v_0/(1+r)$.[9] Then since the support of the stock price distribution is bounded, there exists $N > 0$, such that for all $n > N$, we must have that for all $S \in [a, b]$, $w_{in}(S) \in A$ and $w_{jn}(S) \in B$.

This implies that for all $S \in [a, b]$, $C_i(w_{in}(S)) < \alpha < C_j(w_{jn}(S))$. Now we assert that for all $n > N$ we must have $y_{jn} > 0$. Otherwise suppose for some $n > N$, $y_{jn} \leq 0$.

[9] To disallow negative wealth, we need only to restrict the natural domains of utility functions to be contained in $[0, +\infty)$. This restriction does not have any effect on the proof.

First suppose $y_{jn} = 0$. In this case, $w_i'(S)$ and $w_j'(S)$ are both positive constants. Thus from (4), we have

$$(\frac{1}{\delta_i(S)})' = C_i(w_i(S)) \quad \text{and} \quad (\frac{1}{\delta_j(S)})' = C_j(w_j(S))$$

Since for all $S \in [a, b]$, $C_i(w_{in}(S)) < \alpha < C_j(w_{jn}(S))$, using the fact that $\phi_{in}(S)$ and $\phi_{jn}(S)$ both price the stock correctly, we conclude that $\phi_{in}(S) - \phi_{jn}(S)$ changes its sign twice and will not be zero except for two points. Obviously these two pricing kernels can never agree on the price of the derivative because the one with fatter tails will always give strictly higher prices for convex derivatives.

Now suppose $y_{jn} < 0$. Following the same argument as in the proof for the sufficiency, we conclude that the two pricing kernels cannot agree on the price of the derivative.

Hence for $n > N$ we must have $y_{jn} > 0$. Thus we have a situation where investor j buys the derivative, but investor i does not do so. This completes the proof. Q.E.D.

The above two theorems give an ordering of utility functions in terms of the motive to buy options. This ordering is not complete since not all functions can be ordered in such a way.

Note it is restrictive that we require one investor is strongly more *cautious* than another. The reason that we need this strong condition is because we have to deal with the situation where the investors may have any optimal positions in the stock market and the money market.

4 Numerical Examples

In this section we present some numerical examples to show how an investor's cautiousness measure affects his position in the options market. We use the popular HARA class utility functions to model investors' preferences. As investors are assumed to have HARA class utility functions, their cautiousness measures are nonnegative and constant. Explicitly, an investor's utility function is given as follows:

$$u(x) \equiv \frac{(x + a)^{1-\gamma}}{1 - \gamma},$$

where a and $\gamma > 0$ are both constant. We can verify that the investor's cautiousness measure is equal to $1/\gamma$.

We assume the distribution of the underlying stock price is lognormal. The option is chosen to be the at-the-money call option, i.e., the exercise price of the option is $X = S_0$. In the genetic example, the values of the parameters in Problem (1) are set as follows: $w_0 = 1000, c_0 = 13.27, r = 0.1, S_0 = 100, \mu = 0.15, \sigma = 0.2, a = 0, \gamma = 3$.

In the following tables we present calculations of x and y when the value of substance parameter a changes from -3200 to 3200 for $\gamma = 0.8, 0.9, 1, 1.1$. Note the range of substance parameter a is from negative 3.2 times initial wealth to positive 3.2 times initial wealth, which is quite significant. These calculations demonstrate that however the value of substance parameter a changes, the sign of the investor's position in the options market, y, never changes for a specific value of γ. Note by keeping γ constant while changing a, we change the risk aversion measure while the cautiousness measure remains constant. Hence these calculations emphatically confirm that it is the cautiousness measure not the risk aversion measure determines the sign of the investor's position in the option market.

a **Varies,** $\gamma = 0.8$

a	-3200	-1600	-800	-400	400	800	1600	3200
x	132	156	170	176	190	198	210	236
y	31	38	43	45	46	47	52	60

a **Varies,** $\gamma = 0.9$

a	-3200	-1600	-800	-400	400	800	1600	3200
x	125	151	164	171	184	189	203	230
y	18	20	22	23	23	27	26	27

a **Varies,** $\gamma = 1$

a	-3200	-1600	-800	-400	400	800	1600	3200
x	126	152	165	172	185	191	204	230
y	-1.5	-1.8	-1.9	-2	-2.4	-2.3	-2.5	-2.8

a **Varies,** $\gamma = 1.1$

a	-3200	-1600	-800	-400	400	800	1600	3200
x	123	149	161	168	180	187	199	225
y	-12	-16	-16	-17	-18	-18	-20	-22

In the following table we present the caculations of x and y when γ varies from 0.2 to 10.

γ **Varies,** $a = 0$

γ	0.2	0.4	0.6	0.8	0.9	1	1.2	1.4	1.6	1.8	3	10
x	551	283	193	180	177	178	169	157	146	136	93	32
y	362	187	126	48	23	-2	-28	-39	-45	-48	-43	-18

From the above results, we are clear that the sign of the investor's position in the options market, y, is positive when his cautiousness measure is lager than 1/0.9, and it is negative when his cautiousness measure is larger than 1.

5 Properties of Cautiousness

We have given an ordering of utility functions in terms of their *cautiousness*. Utility functions can be ordered in such a way are of special interest when we compare investors' motive to buy options. Note since HARA class utility functions have constant *cautiousness* thus they are ideal candidates for this purpose. But surely HARA utility functions are not the only utility functions can be ordered in such a way. Nevertheless we will see that this ordering of utility functions is closely related to HARA utility functions. We have the following result.

Proposition 1 *The following two statements are equivalent.*

1. *There exists a constant $C > 0$ such that for any x and y, $C_i(x) \geq C \geq C_j(y)$.*
2. *There exists a constant $C > 0$ such that $u_i'(x) = t_i(x)^{-1/C}$, where $t_i(x)$ is a positive and concave function, and $u_j'(x) = t_j(x)^{-1/C}$, where $t_j(x)$ is a positive and convex function.*

Proof: Let $v(x) = x^{1-\frac{1}{C}}/(1-\frac{1}{C})$. Let $u_i'(x) \equiv v'(t_i(x)) = (t_i(x))^{-\frac{1}{C}}$, where we require $t_i(x)$ to be positive. Thus we have $R_i(x) = \frac{1}{C}\frac{t_i'(x)}{t_i(x)}$. It follows that

$$C_i(x) \equiv (\frac{1}{R_i(x)})' = C - C\frac{t_i(x)t_i''(x)}{t_i'^2(x)}.$$

Hence $C_i(x) \geq C$ is equivalent to $t_i''(x) \leq 0$.

The result about $u_j(x)$ can be proved in the same way. Q.E.D.

Let $u'(x) = t(x)^{-1/C}$, where $C > 0$ and $t(x)$ is positive, we can see that $u(x)$ is concave if and only if $t(x)$ is increasing. Thus in the above proposition if we require utility functions to be concave in the first statement then we require $t_i(x)$ and $t_j(x)$ to be increasing in the second statement.

The above proposition can be used to give examples of $u_i(x)$ and $u_j(x)$ which are non-HARA utility functions such that $u_i(x)$ is strongly more cautious than $u_j(x)$. Note it is easy to find a non-HARA utility function $u_i(x)$ which is DARA and has cautiousness $C_i(x) \geq C \geq 0$: we need only let $u_i'(x) = t_i(x)^{-1/C}$, where $t_i(x)$ is positive, increasing, and concave. But it is not straightforward to find a non-HARA utility function $u_j(x)$ which is DARA and has cautiousness $C_j(x) \leq C > 0$. Thus we give the following result.

Corollary 3 *The following two statements are equivalent.*

1. *Utility function $u_j(x)$ is concave and DARA and there exists a constant $C > 0$ such that for all x, $C_j(x) \leq C$.*
2. *There exists a constant $C > 0$ such that $u_j'(x) = e^{-\int_a^x \frac{1}{C\nu(y)}dy}$, where for all $x > a$, $\nu(x)$ is a positive and increasing function whose first derivative is smaller than unity.*

Proof: From the preceding proposition there exists a constant $C > 0$ such that for all x, $C_j(x) \leq C$ if and only if $u_j'(x) = t_j(x)^{-1/C}$, where $t_j(x)$ is a positive and convex function. Note as $R_j(x) = \frac{1}{C}\frac{t_j'(x)}{t_j(x)}$, we can see that $u_j(x)$ is DARA, i.e., $C_j(x)$ is positive, if and only if $\frac{t_j'(x)}{t_j(x)}$ is decreasing, i.e., $\ln t_j(x)$ is concave. Thus the first statement is true if and only if $u_j'(x) = t_j(x)^{-1/C}$, where $t_j(x)$ is positive, increasing, convex, and log-concave.

But $t_j(x)$ is positive, increasing, convex, and log-concave if and only if $t_j(x) \equiv e^{\kappa(x)}$, where $\kappa(x)$ is increasing and concave and satisfies $\kappa'^2(x)+\kappa''(x) \geq 0$, i.e., $(\frac{1}{\kappa'(x)})' \leq 1$. Hence a concave and DARA utility function $u_j(x)$ has cautiousness $0 \leq C_j(x) \leq C$ if and only if $u_j'(x) = e^{-\kappa(x)/C}$, where $\kappa(x)$ is increasing and concave and satisfies $(\frac{1}{\kappa'(x)})' \leq 1$. It follows that $u_j(x)$ has cautiousness $C_j(x) \leq C$ for all $x > a$ if and only if $u_j'(x) = e^{-\int_a^x \frac{1}{C\nu(y)}dy}$, where for all $x > a$, $\nu(x)$ is a positive and increasing function whose first derivative is smaller than unity. Q.E.D.

Assume there are a set of utility functions ordered by their cautiousness; the question is: do basic operations on utility functions preserve the ordering? We have the following result.

Proposition 2 *The operation $u(x) :\rightarrow u(ax+b)$ preserves the ordering of utility functions.*

Proof: Let $u_1(x)$ and $u_2(x)$ are two of a set of ordered utility functions such that $C_1(x) \geq C \geq C_2(x)$, where $C_i(x)$ is the *cautiousness* of $u_i(x)$, $i = 1, 2$. We have

$$C_i(ax+b) = \frac{(au_i'(ax+b))(a^3u_i'''(ax+b))}{a^4u_i''^2(ax+b)} - 1 = \frac{u_i'(ax+b)u_i'''(ax+b)}{u_i''^2(ax+b)} - 1.$$

It follows that $C_1(ax+b) \geq C \geq C_2(ax+b)$. Hence the ordering is preserved. Q.E.D.

While the above operation completely preserve the ordering, some operations may partially preserve it. For example, we have the following result.

Proposition 3 *If $u_1((x)$, $u_2(x), ..., u_n(x)$ all have cautiousness higher than a constant then the cautiousness of $\sum_1^n a_iu_i(x)$ is also higher than the constant.*

Proof: The general statement follows from the case $u(x) = u_1(x) + u_2(x)$. For this case,

$$C(x) = \frac{(u_1'(x) + u_2'(x))(u_1'''(x) + u_2'''(x))}{(u_1''(x) + u_2''(x))^2} - 1.$$

It follows that

$$C(x) = \frac{(u_1'(x) + u_2'(x))((C_1(x)+1)\frac{u_1''^2(x)}{u'(x)} + (C_2(x)+1)\frac{u_2''^2(x)}{u'(x)})}{(u_1''(x) + u_2''(x))^2} - 1.$$

Suppose $C_i(x) \geq C$, $i = 1, 2$, then

$$C(x) \geq (C+1)\frac{(u_1'(x) + u_2'(x))(\frac{u_1''^2(x)}{u'(x)} + \frac{u_2''^2(x)}{u'(x)})}{(u_1''(x) + u_2''(x))^2} - 1 \geq C.$$

Q.E.D.

We also have the following result.

Proposition 4 *Given utility function $u_1(x)$ and $u_2(x)$, if they both have cautiousness higher than constant $C \leq 0.5$, then $u(x) \equiv u_1(u_2(x))$ also has cautiousness higher than C; if they both have cautiousness lower than constant $C \geq 0.5$, then $u(x) \equiv u_1(u_2(x))$ also has cautiousness lower than C.*

Proof: Let $C(x)$ be the *cautiousness* of $u(x)$. Then

$$C(x) = \frac{u_1'(u_2)u_2'(u_1'''(u_2)u_2'^3 + 3u_1''(u_2)u_2''u_2' + u_1'(u_2)u_2''')}{(u_1''(u_2)u_2'^2 + u_1'(u_2)u_2'')^2} - 1,$$

where for brevity the argument of $u_2(x)$ is omitted. It can be written as

$$C(x) = \frac{(C_1(u_2)+1)u_1''^2(u_2)u_2'^4 + 3u_1''(u_2)u_1'(u_2)u_2'^2u_2'' + (C_2+1)u_1'^2(u_2)u_2''^2}{(u_1''(u_2)u_2'^2 + u_1'(u_2)u_2'')^2} - 1$$

If $C_i(y) \geq C$, $i = 1, 2$, where $C \leq 0.5$, then

$$C(x) \geq (C+1)\frac{u_1''^2(u_2)u_2'^4 + 2u_1''(u_2)u_1'(u_2)u_2'^2u_2'' + u_1'^2(u_2)u_2''^2}{(u_1''(u_2)u_2'^2 + u_1'(u_2)u_2'')^2} - 1 = C$$

If $C_i(y) \leq C$, $i = 1, 2$, where $C \geq 0.5$, then

$$C(x) \leq (C+1)\frac{u_1''^2(u_2)u_2'^4 + 2u_1''(u_2)u_1'(u_2)u_2'^2u_2'' + u_1'^2(u_2)u_2''^2}{(u_1''(u_2)u_2'^2 + u_1'(u_2)u_2'')^2} - 1 = C$$

Q.E.D.

6 Applications

6.1 Impact of Background Risk on Risk-Sharing

We have shown that *cautiousness* measures an investor's motive to buy options. In this section we investigate the impact of background risk on an investor's *cautiousness* hence on his motive to buy options. The case of additive background risk has already been discussed by Nachman (1982), Gollier (2001), and HHK (2006b).

Assume the background risk is additive. Let $P(x)$ and $R(x)$ denote the absolute prudence and absolute risk aversion of the original utility function respectively. Let $\tilde{P}(x)$ and $\tilde{R}(x)$ denote the absolute prudence and absolute risk aversion of the derived utility function respectively. It was showed that if for all x, $P(x) \geq kR(x)$, then for all x, $\tilde{P}(x) \geq k\tilde{R}(x)$, where $k > 0$ is a constant.[10] HHK derived a sufficient condition for the presence of additive background risk to increase an investor's cautiousness. They showed that if an investor has decreasing and convex cautiousness, then his cautiousness will be uniformly higher when exposed to additive background risk.

In this section we considering both additive and multiplicative background risks. Given a utility function, $u(x)$, when there is an additive (multiplicative) background risk ϵ, as usual, we denote the derived utility function by $\hat{u}(x)$. For additive (multiplicative) background risk ϵ we have $\hat{u}(x) \equiv Eu(x + \epsilon)$ ($\hat{u}(x) \equiv Eu(x\epsilon)$). We have the following result.

Proposition 5 *Assume an investor's utility function $u(x)$ has positive third derivative and its cautiousness is higher than a constant. If the investor has an additive or multiplicative background risk or both, the cautiousness of the derived utility function will also be higher than the constant.*

Proof: We denote the background risk as ϵ, a random variable. Assume the *cautiousness* of $u(x)$ is higher than constant C. Let $R(x)$ and $P(x)$ be the risk aversion and prudence of the utility function. Then we have $P(x)/R(x) \geq C+1$.

We prove the result for the case of multiplicative background risk. Other cases can be similarly proved. Note for positive a and b we have $a + b \geq 2\sqrt{ab}$. Thus for any e_1 and e_2,

$$\frac{P(xe_1)e_1}{R(xe_2)e_2} + \frac{P(xe_2)e_2}{R(xe_1)e_1} \geq 2\sqrt{\frac{P(xe_1)P(xe_2)}{R(xe_1)R(xe_2)}} \geq 2(C+1) \tag{9}$$

Rearranging the terms in (9), we have, for any e_1 and e_2,

$$u'''(xe_1)e_1^3u'(xe_2)e_2 + u'''(xe_2)e_2^3u'(xe_1)e_1 \geq 2(C+1)u''(xe_1)e_1^2u''(xe_2)e_2^2 \tag{10}$$

Assuming e_1 and e_2 are independent and have identical distributions as ϵ and taking the expectation of (10) with respect to e_1 and e_2, we obtain

$$2E(u'''(x\epsilon)\epsilon^3)E(u'(x\epsilon)\epsilon) \geq 2(C+1)(E(u''(x\epsilon)\epsilon^2))^2 \tag{11}$$

Rearranging the terms in (11), we have $\hat{P}(x)/\hat{R}(x) \geq C + 1$, where $\hat{R}(x)$ and $\hat{P}(x)$ are the risk aversion and prudence of the derived utility $Eu(x\epsilon)$. Hence the *cautiousness* of the derived utility, $\hat{C}(x) = \hat{P}(x)/\hat{R}(x) - 1 \geq C$. Q.E.D.

Corollary 4 *Assume an investor's utility function is HARA class with positive cautiousness. If the investor has an additive (multiplicative) background risk, the cautiousness of the derived utility function will be higher, and it will be strictly higher unless the utility function is exponential (power) utility. If the investor has both additive and multiplicative risks, the cautiousness of the derived utility function will be strictly higher.*

[10] See, for example, Proposition 23, page 115, Gollier (2001).

Proof: Note that a HARA utility function has constant *cautiousness*, say C. It follows from Proposition 5 that the *cautiousness* of the derived utility function is higher than C. Note in the proof of Proposition 5 the inequalities are strict for additive (multiplicative) background risk unless the utility function has constant absolute (relative) risk aversion, that is, it is exponential (power) utility. Hence if a utility function is HARA class, given an additive (multiplicative) background risk, the *cautiousness* of the derived utility function will be strictly higher unless the utility function is exponential (power) utility. This proves the first result.

If the investor has both additive and multiplicative background risks, from Proposition 5, the cautiousness of his derived utility function must be higher. Moreover, as his utility function can not be both exponential and power utility function, from the first result, the cautiousness of his derived utility function must be strictly higher. Q.E.D.

The above result shows that if an investor has HARA class utility then when he has a background risk, the *cautiousness* of his derived utility function will be higher, hence he will have a stronger motive to buy options.

FSS (1998) also studied the impact of an additive background risk on an investor's demand for options. They showed that in an economy in which investors have identical constant positive *cautiousness* the investors without background risk will have globally concave optimal payoff functions, which they interpreted that an additive background risk makes an investor more likely to buy options.

The difference between Corollary 4 and FSS's main result is worth noting although they give the similar conclusion. FSS's model relies on the assumption that there is a complete market of contingent claims on the stock and the assumption that all investors are expected utility maximizers and have identical *cautiousness* while Corollary 4 does not need these assumptions at all. Note Corollary 4 is even valid when many other investors are not rational utility maximizers.

Corollary 4 can also be used to extend the main result in FSS (1998) to the case where investors have either additive or multiplicative background risk or both. This extension is presented in Appendix A.

6.2 Who Buys More Stocks and Who Buys More Bonds

Consider a firm which has outstanding bonds and stocks. Assume the bonds have no coupons and they all have the same maturity time. Assume both the stocks and the bonds are traded in the open market. Consider investors who invest their money in the firm's bonds and stocks. Assume there is also a money market where they can borrow and lend money at the risk-free interest rate. Thus these investors can form portfolios of the money market instrument, the firm's bonds and stocks. Assume the investors maximize their expected utility of their investments in such portfolios.

Let c_0 and B_0 be the initial total value of the stock and the bond respectively. Let the value of the company at the maturity of the bond be S. Let the face value of the bond be X. Then the total payoff of the stocks at the maturity of the bond is $(S-X)^+$, which implies that the shares are just call options on the firm value. Moreover, the payoff of the bond at maturity is $S-(S-X)^+$. Denote investor i's initial wealth by w_{i0}. Assume he buys x_i fraction of the stock and y_i fraction of the bond and invests the rest of his wealth in the money market. Then the value of his investment at maturity of the bond is

$$w_i(S) = (w_{i0} - x_i c_0 - y_i B_0)(1+r) + x_i(S-X)^+ + y_i(S-(S-X)^+),$$

where r is the risk-free interest rate.

Proposition 6 *Investor i is strongly more cautious than investor j if and only if for all initial wealth and all initial bond and stock prices (such that there are interior solutions to the investors' investment problem (1)), investor j buys more shares than bonds implies investor i buys more shares than bonds and investor i buys less shares than bonds implies investor j buys less shares than bonds.*

Proof: Note the payoff of investor i's investment at maturity can be written as

$$w_i(S) = (w_{i0} - x_i c_0 - y_i B_0)(1+r) + (x_i - y_i)(S-X)^+ + y_i S.$$

Then from Theorems 1 and 2, it follows that investor i is strongly more cautious than investor j if and only if for all initial wealth w_{i0} and w_{j0} and all initial bond and stock prices c_0 and B_0 (such that there are interior solutions to the investors' investment problem), $x_j > y_j$ implies $x_i > y_i$ and $x_i < y_i$ implies $x_j < y_j$, that is, investor j buys more stocks than bonds implies investor i buys more stocks than bonds and investor i buys less stocks than bonds implies investor j buys less stocks than bonds. Q.E.D.

7 Conclusions

In this paper we have shown that *cautiousness*, which is equivalent to the ratio of prudence to risk aversion, is a measure of an investor's motive to buy options. We have shown that if investor i is strongly more *cautious* than j, then investor j buys an option only if i does so, and investor i sells the option only if j does so, regardless of their initial wealth, the underlying stock price, and the option price; and the reverse is also true. The result remains true if the option is replaced by any derivative with a convex payoff.

This result can have many applications in financial markets. For example, as we have shown, it can be used to explain the impact of background risk on demand for options: background risk will increase the *cautiousness* of HARA class utility, thus increase an investor's motive to buy options. In another example, we have also shown that the result can also be used to explain who buys more stocks and who buys more corporate bonds: roughly speaking, an investor with a higher cautiousness measure tends to buy more stocks than bonds.

It is interesting to see that the measure of an investor's motive to buy options is closely related to both measures of risk aversion and prudence which explain investors' activities in the bond market and stock market. Regarding the latter two activities, it is now widely accepted that investors should have decreasing absolute risk aversion while Kimball (1993) proposed decreasing absolute prudence. HHK (2007) showed that when all investors have constant cautiousness then the representative investor has increasing cautiousness. This shows that increasing cautiousness is associated with reasonable preferences. Huang (2000) showed that increasing (decreasing) *cautiousness* implies decreasing (increasing) relative risk aversion (if the utility function satisfies the Lower Inada condition)[11].

According to the model in this paper, an investor's *cautiousness* can tell if he has a stronger motive to buy options than others. However, if two investors both buy options, their coefficients of *cautiousness* can not tell if one buys more options than the other.

Moreover, although the model in this paper shows *cautiousness* tells who buys options but it does not tell what the options are bought for: they may be used to hedge the risk in the underlying stock; they may be also used for their leverage effect. Thus *cautiousness* is not necessarily a measure of cautiousness.

Finally, although we have shown that background risk will increase the *cautiousness* of HARA class utility, the impact of background risk on the *cautiousness* of a general utility function is still unclear. Further research on this is needed.

[11] A utility function $u(x)$ is said to satisfy the Lower Inada Condition if $\lim_{x \to 0} u'(x) = +\infty$.

Appendix A Background Risk and Concave Sharing Rules

We use a model which is similar to that used by FSS (1998). Assume in a one-period economy there are N investors and every investor's wealth consists of a portfolio of state-contingent claims on the market portfolio. Let X be the payoff of the market portfolio at the end of the period.

Assumption 1 *Assume that there is a complete market for state-contingent claims on X.*

This assumption ensures that all investors can buy and sell state-contingent claims on X so that, as discussed in Leland (1980), any investor i can choose a risk-sharing rule $x_i(X)$.

Assumption 2 *Assume all investors are rational expected-utility-maximizers.*

This assumption will ensure that we can derive a pricing representative investor whose preference determines the price of contingent claims. As is well known, the pricing representative investor's preference will be reflected in the unique pricing kernel, $\phi(X)$, which is determined in the equilibrium of the economy.

Note the difference between this model and the model used in Section 2 to develop the measure of an investor's motive to buy options. There we do not need the above two assumptions.

Let $u_i(x)$ denote investor i's utility function. Let w_{i0} be investor i's initial endowment, expressed as the fraction of the spot value of the total wealth in the economy. Let x_i be his optimal payoff function respectively. Then the investor has the following utility maximization problem:

$$\max_{x_i} Eu_i(x_i). \tag{12}$$

Subject to

$$E(\phi x_i) = w_{i0} E(\phi X). \tag{13}$$

where $E(.)$ denotes the expectation operator. In equilibrium, the market is cleared, thus we have

$$\sum_i x_i(X) = X. \tag{14}$$

We have the first order condition

$$u_i'(x_i) = \lambda_i \phi(X). \tag{15}$$

Differentiating both sides of (15) will lead to the following result:

$$x_i'(X) = R_e(X)/R_i(x_i), \tag{16}$$

where $R_i(x) = -u_i''/u_i'(x)$ is investor i's absolute risk aversion and $R_e(X) = -\phi'(X)/\phi(X)$ is the pricing representative investor's absolute risk aversion.

Differentiating both sides of (16), we obtain

$$x_i''(X) = R_e^2(X)[C_i(x_i) - C_e(X)]/R_i(x_i), \tag{17}$$

where $C_i(x)$ is investor i's *cautiousness* and $C_e(X) = (1/R_e(X))'$ is the pricing representative investor's *cautiousness*.

From (14) and (16) we obtain

$$R_e(X) = (\sum_i R_i^{-1}(x_i))^{-1} \tag{18}$$

From (14) and (17), we obtain

$$C_e(X) = \sum_i s_i C_i(x_i). \tag{19}$$

where $s_i = R_i^{-1}(x_i)/\sum_i R_i^{-1}(x_i)$.

We have the following result.

Proposition 7 *Assume all investors have identical non-negative constant cautiousness* and some investors have independent uninsurable additive and/or multiplicative background risk. Then the risk-sharing rules of investors without background risk are concave, and they are strictly concave unless either all investors have power utility and those with background risk have multiplicative background risk only or all investors have exponential utility and those with background risk have additive background risk only.

Proof: Assume all investors have identical positive constant *cautiousness* C. When investor i is exposed to independent additive and multiplicative background risk ϵ_{i1} and ϵ_{i2}, the utility function $u_i(x_i)$ in the utility maximization problem (12) is replaced by the indirect utility function $\hat{u}_i(x_i) \equiv E(u_i(x_i\epsilon_{i1} + \epsilon_{i2}))$. Thus on the right hand side of (19) $C_i(x_i)$ is replaced by the *cautiousness* of investor i's derived utility function, $\hat{C}_i(x_i)$, if investor i has background risk. For the investors without background risk, $C_i(x_i) = C$ is a positive constant.

From Corollary 4, we know that for every investor i who has background risk, $\hat{C}_i(x_i) > C$. Thus from Equation (19), we easily verify that the pricing representative investor's *cautiousness* is higher than those of the investors without background risk. From (17) the risk-sharing rules of those without background risk are concave. Moreover, unless all those with multiplicative background risk have power utility and all those with additive background risk have exponential utility, it is easy to see that the pricing representative investor's *cautiousness* is strictly higher than C, the cautiousness of investors without background risk. In this case, the risk-sharing rules of investors without background risk are strictly concave. Q.E.D.

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