The Pricing and Hedging of Interest-Rate Derivatives: Theory and Practice

Ser-Huang Poon\textsuperscript{1}, Richard C. Stapleton\textsuperscript{2} and Marti G. Subrahmanyam \textsuperscript{3}

April 28, 2005

\textsuperscript{1}Manchester Business School
\textsuperscript{2}Manchester Business School
\textsuperscript{3}Stern School of Business
1 Definition of Interest-Rate Derivative Contracts

In this section we provide definitions (using cash flow diagrams) of the basic vanilla interest-rate derivatives. These can be classified as single period (FRA, caplet, Libor futures, option on Libor futures) and as multi-period (swaps, caps, swaptions).
Forward Rate Agreement (FRA)

A forward rate agreement (FRA) is an agreement to exchange fixed-rate interest payments at a rate $k$ for Libor payments, on a principal amount $A$ for the loan period $T$ to $T + \delta$.

The time scale for payments on a $T$-maturity FRA is shown below:

\[
\begin{array}{c}
\text{contract} & \downarrow & T & \downarrow & T + \delta \\
\uparrow & \text{settlement} & \uparrow & \text{loan maturity}
\end{array}
\]

Here, $t = 0$ is the contract agreement date, $T$ is the settlement date for the contract, and $T + \delta$ is the date on which the notional loan underlying the FRA is repaid.

- $A$ is the principal of the underlying loan
- $i_t$ is the spot Libor interest rate at time $t$
- $k$ is the strike rate of the contract
- $\delta$ is the loan period
**Interest-Rate Caplet**

An interest-rate caplet (floorlet) is an option to enter a long (short) FRA at time $T$ at a fixed rate $k$.

The time scale for payments on a $T$-maturity caplet is shown below:

\[
\underbrace{A_{\text{max}}(i_T-k,0)\delta}_{1+i_T}\]

Here, $t = 0$ is the contract agreement date, $T$ is the settlement date, and $T + \delta$ is the date on which the notional loan underlying the FRA is repaid.

- $A$ is the principal of the underlying loan
- $i_t$ is the spot Libor interest rate at time $t$
- $k$ is the strike rate of the contract
- $\delta$ is the loan period
Interest-Rate Derivatives

Interest-Rate Futures

A Libor Futures contract is an agreement made at time $t = 0$ to pay or receive the difference between $H_{t,T}$, the futures price at time $t$ and the price at time $t + 1$, $H_{t+1,T}$, daily until the maturity of the contract. The daily cash flows on a $T$-maturity long futures, contracted at $t = 0$ at a futures price $H_{0,T}$ is shown below:

$+(H_{1,T} - H_{0,T})A\delta + (H_{t+1,T} - H_{t,T})A\delta + (H_{T,T} - H_{T-1,T})A\delta$

\[ \begin{align*}
\downarrow & \downarrow & \downarrow \\
0 & 1 & \ldots & \ldots & t & t+1 & \ldots & T \\
\uparrow & \uparrow & & & & & & \\
\text{contract} & \text{mark-to-market dates} & \text{futures maturity}
\end{align*} \]

- $H_{t,T}$ is the Libor futures price at time $t$
- $A$ is the principal of the contract
- $\delta$ is the loan period
Options on Interest-Rate Futures

A Libor Futures option is an option to enter an interest-rate futures contract at a fixed rate \( k \).

A European-style futures call option with maturity \( T \) has a payoff:

\[
A \max(H_{T,T} - k, 0)\delta
\]

- \( H_{T,T} \) is the Libor futures price at time \( T \)
- \( A \) is the principal of the contract
- \( \delta \) is the loan period
- \( k \) is the strike rate
Interest-Rate Swap

An interest-rate swap is an agreement made at time 0 to exchange fixed-rate interest payments at a rate $k$ for Libor payments, on a principal amount $A$ every $\delta$ years, over the loan period 0 to $n$.

$$
\begin{align*}
+A(i_0 - k)\delta & +A(i_\delta - k)\delta & +A(i_{n-\delta} - k)\delta \\
\downarrow & \downarrow & \downarrow \\
0 & \delta & 2\delta & \cdots & n \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
contract & \text{reset date} & \text{reset date} & \text{swap maturity}
\end{align*}
$$

- $A$ is the principal of the underlying loan
- $i_t$ is the Libor interest rate at time $t$
- $k$ is the strike rate of the contract
- $\delta$ is the loan period
Swaption

A European-style swaption, with strike rate \( k \), gives the right to enter an \( n \)-year swap on the option maturity date \( T \). The cash flows on a pay-fixed swaption (payer swaption) are shown below.

\[
\begin{align*}
\text{if } s_T &> k, & +A(i_T - k)\delta &+A(i_{T+\delta} - k)\delta &+A(i_{T+n-\delta} - k)\delta \\
0 &\quad T & T+\delta & T+2\delta & \ldots & T+n \\
& \quad \uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow \\
& \quad \text{option maturity} & \quad \text{reset date} & \quad \text{reset date} & \quad \text{swap maturity}
\end{align*}
\]

- \( s_T \) is the swap rate at time \( T \)
- \( A \) is the principal of the underlying loan
- \( i_t \) is the Libor interest rate at time \( t \)
- \( k \) is the strike rate of the contract
- \( \delta \) is the loan period
Interest-Rate Cap

An interest-rate cap, with strike rate $k$, gives the right to enter a series of FRAs every $\delta$ years over $n$ years.

\[ +\text{Max}(i_{\delta} - k, 0)\delta \quad \ldots \quad +\text{Max}(i_{n-\delta} - k, 0)\delta \]

\[ 0 \quad \delta \quad 2\delta \quad 3\delta \quad \ldots \quad n \]

\[ \uparrow \quad \uparrow \quad \uparrow \quad \text{reset date} \quad \text{reset date} \quad \uparrow \quad \text{cap end} \]

- $A$ is the principal of the underlying loan
- $i_t$ is the Libor interest rate at time $t$
- $k$ is the strike rate of the contract
- $\delta$ is the loan period

An interest-rate cap is equivalent to a portfolio of put options on one-period zero-coupon bonds.
2 The Valuation of Interest-Rate Derivative Contracts

The definitions in the previous section show the cash flows given that the derivative is contracted at date $t = 0$. In this section we value these products at a date $t$, which as a special case could be $t = 0$. The FRAs and swaps are valued using standard discounting techniques. The options are valued using different versions of the Black model.
Valuation of an FRA

1. Long (time $t = 0$ contract) FRA Payoff

\[ FRA_{t,T,\delta}(k) = A(f_{t,T} - k)\delta B_{t,T+\delta} \]

- $f_{t,T}$ is the Libor forward rate at time $t$ for delivery at $T$
- $B_{t,T+\delta}$ is the price at $t$ of a bond paying $\$1$ at $T + \delta$
Valuation of an Interest-Rate Swap

Assuming that the valuation date \( t \) is \( 0 < t < \delta \):

**Swap Reversed Cash Flows**

\[
\begin{align*}
0 & \quad t & \quad \delta & \quad 2\delta & \quad 3\delta \\
\uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow \\
\text{contract} & \quad \text{valuation date} & \quad \text{reset date} & \quad \text{reset date} & \quad \text{reset date}
\end{align*}
\]

\[
\begin{align*}
+A(i_0 - k)\delta & \quad +A(f_{t,2\delta} - k)\delta & \quad +A(f_{t,3\delta} - k)\delta \\
\downarrow & \quad \downarrow & \quad \downarrow
\end{align*}
\]

**Swap: Value at Time \( t \)**

\[
\text{swap}_{t,0,n,\delta}(k) =
\begin{align*}
A(i_0 - k)\delta B_{t,\delta} \\
+A(f_{t,2\delta} - k)\delta B_{t,2\delta} \\
+A(f_{t,3\delta} - k)\delta B_{t,3\delta}
\end{align*}
\]

\[
\begin{align*}
0 & \quad t & \quad \delta & \quad 2\delta & \quad 3\delta \\
\uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow & \quad \uparrow \\
\text{contract} & \quad \text{valuation date} & \quad \text{reset date} & \quad \text{reset date} & \quad \text{reset date}
\end{align*}
\]

- \( \text{swap}_{t,0,n,\delta}(k) \) is the value at \( t \) of an \( n \)-year \( \delta \)-reset swap contracted at time \( t = 0 \) at a strike rate \( k \)
- \( f_{t,T} \) is the Libor forward rate at time \( t \) for delivery at \( T \)
- \( B_{t,i\delta} \) is the price at \( t \) of a bond paying $1 at \( i\delta \)
The Black Model for a Caplet

If the BGM process for forward rates holds, the value of a caplet at time $t$, with maturity $T$ is given by:

$$caplet_{t,T,\delta}(k) = \frac{A}{1 + f_{t,T}^\delta} \left[ f_{t,T} N(d_1) - k N(d_2) \right] B_{t,T}$$  \hspace{1cm} (1)

where

$$d_1 = \frac{\ln\left(\frac{f_{t,T}}{k}\right) + \frac{\sigma^2 \tau}{2}}{\sigma \sqrt{\tau}}$$

$$d_2 = d_1 - \sigma \sqrt{\tau}$$

$$\tau = T - t$$

- $caplet_{t,T,\delta}(k)$ is the value at $t$ of a $T$-year caplet at a strike rate $k$
- $f_{t,T}$ is the Libor forward rate at time $t$ for delivery at $T$
- $B_{t,T}$ is the price at $t$ of a bond paying $\$1$ at $T$
- $\sigma$ is the volatility of the Libor, $i_T$
The Black Model for Libor Futures Options

Assuming these are European-style, marked-to-market options, then a put on the futures price has a futures value

\[ P_{t,T}(k) = [(1 - H_{t,T})N(d_1) - (1 - K)N(d_2)] \]

where

\[ d_1 = \ln \left( \frac{1 - H_{t,T}}{1 - K} \right) + \frac{\sigma^2 \tau}{2} \]
\[ d_2 = d_1 - \sigma \sqrt{\tau} \]

where \( H_{t,T} \) is the futures price and \( K \) is the strike price.

The futures price of the option can be established for Libor options by assuming that the futures rate follows a lognormal diffusion process (limit of the geometric binomial process as \( n \to \infty \)).

- \( P_{t,T}(k) \) is the value at \( t \) of a \( T \)-year futures option at a strike rate \( k \)
- \( H_{t,T} \) is the Libor futures price at time \( t \) for delivery at \( T \)
- \( \sigma \) is the volatility of Libor
The Black Model for Caplets/Floorlets using Equivalent Bond Options

If the (continuously compounded) interest rate is normally distributed, the price of a floorlet on Libor is given by

$$\text{Floorlet}_{t,T,\delta}(k) = [B_{t,T+\delta}N(d_1) - KB_{t,T}N(d_2)](1 + k\delta)$$

where

$$d_1 = \frac{\ln \left( \frac{F_{t,T,T+\delta}}{K} \right) + \frac{\sigma'^2 \tau}{2}}{\sigma' \sqrt{\tau}}$$

$$d_2 = d_1 - \sigma' \sqrt{\tau}$$

$$K = \frac{1}{1 + k\delta}$$

$$\tau = T - t$$

Here, the bond forward price is assumed to follow a lognormal diffusion process.

- \text{floorlet}_{t,T,\delta}(k) is the value at } t \text{ of a } T\text{-year floorlet at a strike rate } k
- B_{t,T+\delta} \text{ is the price at } t \text{ of a bond paying } $1 \text{ at } T + i\delta
- \sigma' \text{ is the volatility of the zero-coupon bond price}
- K \text{ is the strike price of the equivalent bond option}
3 Parity Relationships for Interest-Rate Options

Notation

- $P_t$ value of put option at time $t$
- $C_t$ value of call option at time $t$
- $K$ strike price
- $T$ option maturity

A long FRA at $k$ is equivalent to $(1 + k\delta)$ short forward contracts on a one-period zero-coupon bond.

A caplet on Libor at $k$ is equivalent to $(1 + k\delta)$ put options on a one-period zero-coupon bond.

An long interest-rate swap (to pay fixed, receive Libor) is equivalent to a short forward contract on an $n$-year coupon bond, with coupon $k$.

A European-style payer swaption (to pay fixed, receive Libor) is equivalent to a put option on an $n$-year coupon bond, with coupon $k$, and with a strike price of par.