Foundations of Finance Theory: Course Notes

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1 Mean-variance portfolio analysis and the CAPM

Portfolio Demand and the CAPM

Assume

- $j = 1, 2, ..., J$ **firms in the economy, with** N_j **shares outstanding. 100% equity. Cash flow,** x_j **at time** $t = 1$
- $i = 1, 2, ..., I$ **investors with wealth** $w_{0,i}$ **at** $t = 0$ **and utility**

$$u_i = u_i[E(w_i), var(w_i)],$$

where $w_i \equiv w_{1,i}$ **is wealth at time** $t = 1$

- **Investors have homogeneous expectations**
- **Investors can borrow or lend at risk-free rate of interest** r_f
- **No taxes, transactions costs**

Definitions

Let:

- $S_j = s_j N_j$ **be market capitalisation of** j, s_j **is price of 1 share**
- $n_{i,j}$ **shares bought by investor** i **in firm** j
- $$\alpha_{i,j} = \frac{n_{i,j}}{N_j}$$
- B_i **is bond investment of investor** i
- $\sigma_{j,j} = var(x_j)$
- $\sigma_{j,k} = covar(x_j, x_k)$

Hence

$$w_i = \sum_j \alpha_{i,j} x_j + B_i(1 + r_f) \quad (1)$$

where

$$B_i = w_{0,i} - \sum_j \alpha_{i,j} S_j$$

Portfolio Demand

Investor i chooses shares $n_{i,j}$ to maximise utility:

$$max\ u_i = u_i[E(w_i), var(w_i)] \qquad (2)$$

where

$$w_i = \sum_j \alpha_{i,j} x_j + B_i(1 + r_f)$$

subject to

$$B_i + \sum_j \alpha_{i,j} S_j = w_{0,i}$$

Step 1: Compute Portfolio Mean and Variance

Mean:

$$E(w_i) = w_{0,i}(1+r_f)+\sum_j \alpha_{i,j}[E(x_j)-S_j(1+r_f)] \tag{3}$$

Variance:

$$\begin{aligned} var(w_i) &= var(\sum_j \alpha_{i,j}x_j) \\ &= \alpha_{i,1}^2\sigma_{1,1} + 2\alpha_{i,1}\alpha_{i,2}\sigma_{1,2} + ... \\ &+ 2\alpha_{i,1}\alpha_{i,j}\sigma_{1,j} + \alpha_{i,2}^2\sigma_{2,2} + ... \\ &+ \alpha_{i,j}^2\sigma_{j,j} \end{aligned} \tag{4}$$

Hence

$$\frac{\partial E(w_i)}{\partial \alpha_{i,j}} = E(x_j) - S_j(1 + r_f) \tag{5}$$

$$\begin{aligned} \frac{\partial var(w_i)}{\partial \alpha_{i,j}} &= 2\alpha_{i,1}\sigma_{j,1} + 2\alpha_{i,2}\sigma_{j,2} + ... + 2\alpha_{i,j}\sigma_{j,j} \\ &+ ... \end{aligned} \tag{6}$$

Step 2: First Order Conditions

Hence

$$\frac{\partial u_i}{\partial \alpha_{i,j}} = \frac{\partial u_i}{\partial E(w_i)}\frac{\partial E(w_i)}{\partial \alpha_{i,j}} + \frac{\partial u_i}{\partial var(w_i)}\frac{\partial var(w_i)}{\partial \alpha_{i,j}} = 0, \forall j$$

or

$$\frac{-\frac{\partial u_i}{\partial E(w_i)}}{\frac{\partial u_i}{\partial var(w_i)}}\frac{\partial E(w_i)}{\partial \alpha_{i,j}} = \frac{\partial var(w_i)}{\partial \alpha_{i,j}}, \forall j$$

Substituting (5) and (6)

$$\lambda_i[E(x_i) - S_j(1 + r_f)] = \alpha_{i,1}\sigma_{j,1} + \alpha_{i,2}\sigma_{j,2} + ... + \alpha_{i,j}\sigma_{j,j} + ..., \forall j$$

where

$$\lambda_i = \frac{-\frac{\partial u_i}{\partial Ew_i}}{2\frac{\partial u_i}{\partial var(w_i)}}$$

Step 3: Optimal Stock Proportions

This set of simultaneous equations can be written in matrix form:

$$\lambda_i[E(x) - S(1 + r_f)] = A\alpha_i$$

where

$$E(x) = \begin{pmatrix} E(x_1) \\ E(x_2) \\ . \\ . \\ E(x_J) \end{pmatrix}, S = \begin{pmatrix} S_1 \\ S_2 \\ . \\ . \\ S_J \end{pmatrix}, \alpha_i = \begin{pmatrix} \alpha_{i,1} \\ \alpha_{i,2} \\ . \\ . \\ \alpha_{i,J} \end{pmatrix}$$

and

$$A = \begin{pmatrix} \sigma_{1,1}, \sigma_{1,2}, ..., \sigma_{1,J} \\ \sigma_{2,1}, \sigma_{2,2}, ..., \sigma_{2,J} \\ \sigma_{3,1}, \sigma_{3,2}, ..., \sigma_{3,J} \\ . \quad . \quad . \\ \sigma_{J,1}, \sigma_{J,2}, ..., \sigma_{J,J} \end{pmatrix}$$

The solution is

$$\alpha_i = \lambda_i A^{-1}[E(x) - S(1 + r_f)] \qquad (7)$$

(7) proves the mutual fund *Separation Theorem*

Step 4: Market Equilibrium In equilibrium all stocks must be held, or

$$\sum_i \alpha_{i,j} = 1, \forall j$$

Summing (7) over i

$$\sum_i \alpha_i = \sum_i \lambda_i A^{-1}[E(x) - S(1 + r_f)] = \overline{1},$$

where

$$\overline{1} = \begin{pmatrix} 1 \\ 1 \\ . \\ . \\ \\ 1 \end{pmatrix}$$

Solving for S

$$S = \frac{1}{1 + r_f}\left(E(x) - \frac{A\overline{1}}{\lambda}\right), \qquad (8)$$

where

$$A\overline{1} = \begin{pmatrix} \sigma_{1,1} + \sigma_{1,2} + \sigma_{1,3} + ... + \sigma_{1,J} \\ \sigma_{2,1} + \sigma_{2,2} + \sigma_{2,3} + ... + \sigma_{2,J} \\ . \qquad . \qquad . \\ \sigma_{J,1} + \sigma_{J,2} + \sigma_{J,3} + ... + \sigma_{J,J} \end{pmatrix}$$

Step 5: The CAPM

For stock 1:

$$\begin{aligned}\sigma_{1,1} &+ \sigma_{1,2} + \sigma_{1,3} + ... + \sigma_{1,J} = cov(x_1, x_1) \\ &+ cov(x_1, x_2) + ... + cov(x_1, x_J) \\ &= cov\left(x_1, \sum_{j=1}^{J} x_j\right) \\ &= cov\left(x_1, x_m\right),\end{aligned}$$

where

$$x_m = \sum_{j=1}^{J} x_j$$

is the *market cash flow***.**

$$S_j = \frac{1}{1 + r_f}\left[Ex_j - \frac{1}{\lambda}cov(x_j, x_m)\right], \tag{9}$$

Step 6: The CAPM: Expected Return

Divide (9) by S_j,

$$1 = \frac{1}{1+r_f}\left[\frac{E(x_j)}{S_j} - \frac{1}{\lambda} cov\left(\frac{x_j}{S_j}, x_m\right)\right],$$

Define the return:

$$r_j = \frac{x_j - S_j}{S_j},$$

$$1 = \frac{1}{1+r_f}\left[E(r_j) + 1 - \frac{S_m}{\lambda} cov\left(\frac{x_j}{S_j}, \frac{x_m}{S_m}\right)\right],$$

$$E(r_j) = r_f + \frac{S_m var(r_m)}{\lambda}\frac{cov(r_j, r_m)}{var(r_m)},$$

$$E(r_j) = r_f + \lambda' \beta_j.$$

Step 7: The CAPM: Closed Form Solution

CAPM holds also for portfolios of shares

If CAPM holds for the market portfolio:

$$E(r_m) = r_f + \lambda' \beta_m.$$

But, $\beta_m = 1$, hence

$$E(r_m) = r_f + \lambda'.$$

The market risk premium:

$$E(r_m) - r_f = \lambda'.$$

and hence

$$E(r_j) = r_f + [E(r_m) - r_f]\beta_j.$$

Optimal Portfolios: Numerical Example

Mean-variance portfolio analysis and the CAPM

Data

j	$E(x_j)$	S_j	σ_j	$\rho_{1,2}$	N_j
1	9	7.5	1.5	0.5	1500
2	38	34	2		4000

i	λ_i	$w_{0,i}$
1	0.3	30000

$$r_f = 0.05$$

Optimal portfolio

Step 1: Compute portfolio mean and variance

Mean

$$E(x_j) - S_j(1 + r_f)$$

$$j = 1: \quad 9 - 7.5(1.05) = 1.125$$

$$j = 2: \quad 38 - 34(1.05) = 2.3$$

$$E(w_i) = 30000(1.05) + \alpha_{i,1}(1.125) + \alpha_{i,2}(2.3)$$

$$\frac{\partial E(w_i)}{\partial \alpha_{i,1}} = 1.125$$

$$\frac{\partial E(w_i)}{\partial \alpha_{i,2}} = 2.3$$

Variance

$$var(w_i) = \alpha_{i,1}^2 \sigma_{1,1} + 2\alpha_{i,1}\alpha_{i,2}\sigma_{1,2} + \alpha_{i,2}^2 \sigma_{2,2}$$

$$\sigma_{1,2} = \sigma_1 \sigma_2 \rho_{1,2} = 1.5(2)(0.5) = 1.5$$

$$\sigma_{1,1} = \sigma_1^2 = 1.5^2 = 2.25$$
$$\sigma_{2,2} = \sigma_2^2 = 2^2 = 4$$

$$var(w_i) = \alpha_{i,1}^2 2.25 + 2\alpha_{i,1}\alpha_{i,2}1.5 + \alpha_{i,2}^2 4$$
$$\frac{\partial var(w_i)}{\partial \alpha_{i,1}} = 2\alpha_{i,1}2.25 + 2\alpha_{i,2}1.5$$
$$\frac{\partial var(w_i)}{\partial \alpha_{i,2}} = 2\alpha_{i,1}1.5 + 2\alpha_{i,2}4$$

Step 2: First order conditions

$$\lambda_i[E(x_j) - S_j(1 + r_f)] = \sigma_{j,1}\alpha_{i,1} + \sigma_{j,2}\alpha_{i,2}$$

$$j = 1 : 0.3(1.125) = 2.25\alpha_{i,1} + 1.5\alpha_{i,2}$$
$$j = 2 : 0.3(2.3) = 1.5\alpha_{i,1} + 4\alpha_{i,2}$$

Step 3: Matrix form

$$\boldsymbol{\lambda}_i[\boldsymbol{E}(\boldsymbol{x}) - \boldsymbol{S}(1 + \boldsymbol{r}_f)] = \boldsymbol{A}\boldsymbol{\alpha}_i$$

$$\boldsymbol{A} = \begin{pmatrix} \mathbf{2.25} & \mathbf{1.5} \\ \mathbf{1.5} & \mathbf{4} \end{pmatrix}$$

$$\mathbf{0.3}\begin{pmatrix} \mathbf{1.125} \\ \mathbf{2.3} \end{pmatrix} = \begin{pmatrix} \mathbf{2.25} & \mathbf{1.5} \\ \mathbf{1.5} & \mathbf{4} \end{pmatrix}\begin{pmatrix} \boldsymbol{\alpha}_{i,1} \\ \boldsymbol{\alpha}_{i,2} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{0.3375} \\ \mathbf{0.69} \end{pmatrix} = \begin{pmatrix} \mathbf{2.25} & \mathbf{1.5} \\ \mathbf{1.5} & \mathbf{4} \end{pmatrix}\begin{pmatrix} \boldsymbol{\alpha}_{i,1} \\ \boldsymbol{\alpha}_{i,2} \end{pmatrix}$$

Hence

$$\begin{pmatrix} \boldsymbol{\alpha}_{i,1} \\ \boldsymbol{\alpha}_{i,2} \end{pmatrix} = \begin{pmatrix} \mathbf{2.25} & \mathbf{1.5} \\ \mathbf{1.5} & \mathbf{4} \end{pmatrix}^{-1}\begin{pmatrix} \mathbf{0.3375} \\ \mathbf{0.69} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{\alpha}_{i,1} \\ \boldsymbol{\alpha}_{i,2} \end{pmatrix} = \begin{pmatrix} \mathbf{0.5926} & \mathbf{-0.222} \\ \mathbf{-0.222} & \mathbf{0.333} \end{pmatrix}\begin{pmatrix} \mathbf{0.3375} \\ \mathbf{0.69} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{\alpha}_{i,1} \\ \boldsymbol{\alpha}_{i,2} \end{pmatrix} = \begin{pmatrix} \mathbf{0.0467} \\ \mathbf{0.155} \end{pmatrix}$$

Optimal Share allocation

$$n_{i,1} = \alpha_{i,1}(N_1)$$

$$n_{i,1} = 0.0467(1500) = 70$$

$$n_{i,2} = \alpha_{i,2}(N_2)$$

$$n_{i,2} = 0.69(4000) = 620$$

Stock price

$$s_1 = 7.5/1500 = 0.005$$

$$s_2 = 34/4000 = 0.0085$$

Investment:

stocks

$$n_1 s_1 + n_2 s_2$$

$$70(0.005) + 620(0.0085) = 5.62$$

bonds

$$w - 5.62 = 4.32$$

total 10.00

2 Asset pricing: a complete markets model

Financial Theory

Assets:

Value of Cash Flows

CAPM

Options:

Black-Scholes

Multi-period

Models

Forward / Futures

Prices: Assets/Options

A General Model
Initial setup and key assumptions

1. **Assume a single period from t to $t + T$.**
2. **Assume forward parity holds.**
3. **Assume that there are a finite number of states of the world at time $t + T$**

   A state-contingent claim on state i is defined as a security which pays \$1 if and only if state i occurs.
4. **Assume markets are complete.**
5. **Assume homogeneous expectations**
6. **Assume that the price of a portfolio or package of contingent claims is equal to the sum of the prices of the individual state-contingent claims.**

1: Single Period Model

0——————————————1

t——————————————- $t+T$

- **Nothing happens between t and $t+T$**
- **Dividends are paid either at t or at $t+T$**
- **No trading between t and $t+T$**

2: Forward Parity

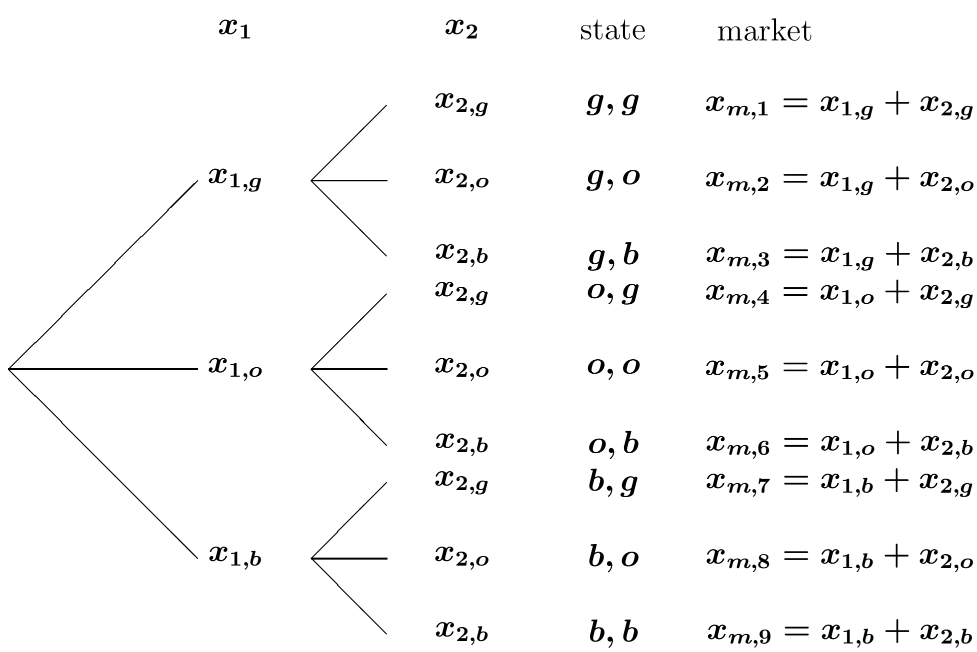
Assume no dividend paid between t and $t + T$

Synthetic forward strategy:

1. **Borrow S_t (price of stock)**
2. **Buy stock**
3. **Repay $S_t(1 + r)$ at $t + T$**

- **Net cash flow at t is zero.**
- **Net cash at $t+T$ is $S_{t+T} - S_t(1+r)$**
- **$S_t(1+r)$ must be forward price**
- $\frac{1}{(1+r)} = B_{t,t+T}$
- $F_{t,t+T} = \frac{S_t}{B_{t,t+T}}$

3: Finite State Space

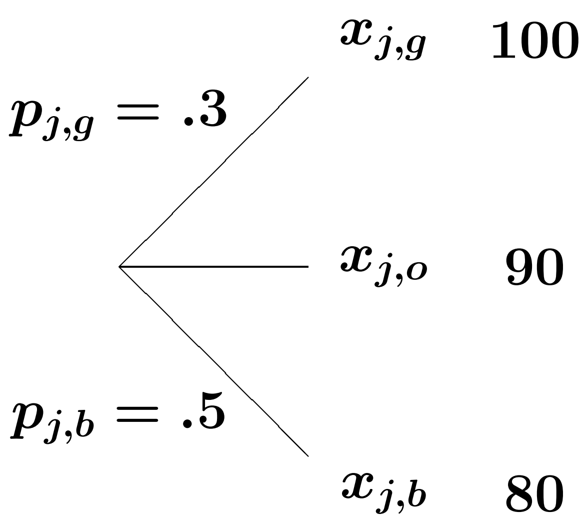


4: Complete Market

Buy (or construct) state contingent claims

- **Buy an option paying \$1 if $x \geq k_1$**
- **Sell an option paying \$1 if $x \geq k_2$**
- **Portfolio pays \$1 if $k_1 < x < k_2$**

5: Homogeneous Expectations



Firm j

- **Investors agree on cash flows, x_j**
- **Investors agree on probabilities, p_j**

6: Asset Forward Price

Law of One price

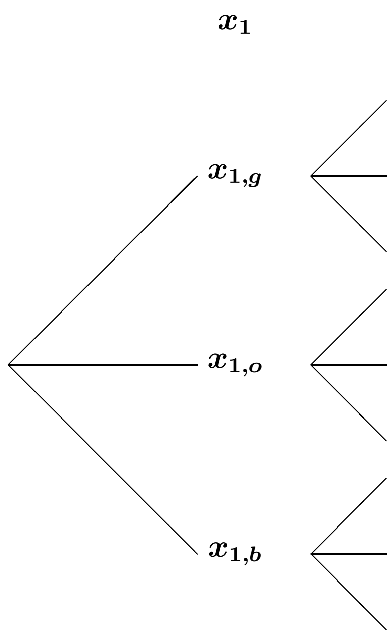
$$F_{j,t,t+T} = \sum_i (x_{j,t+T,i}\ q_i)$$

$$F_j = \sum_i (q_i x_{j,i})$$

Properties of the state price

1. **The state price, q_i, is always greater than zero.**
2. **The state prices sum to 1, i.e. $\Sigma_i q_i = 1$.**

State Price, Pricing Kernel



x_m	state price	prob	pricing kernel
$x_{m,1}$	q_1	p_1	ϕ_1
$x_{m,2}$	q_2	p_2	ϕ_2
$x_{m,3}$	q_3	p_3	ϕ_3
$x_{m,4}$	q_4	p_4	ϕ_4
$x_{m,5}$	q_5	p_5	ϕ_5
$x_{m,6}$	q_6	p_6	ϕ_6
$x_{m,7}$	q_7	p_7	ϕ_7
$x_{m,8}$	q_8	p_8	ϕ_8
$x_{m,9}$	q_9	p_9	ϕ_9

$$\begin{aligned} F_1 &= x_{1,g}(q_1 + q_2 + q_3) \\ &+ x_{1,o}(q_4 + q_5 + q_6) \\ &+ x_{1,b}(q_7 + q_8 + q_9) \end{aligned}$$

The Pricing Kernel, ϕ_i

$$\phi_i = \frac{q_i}{p_i},$$

The properties of ϕ_i are as follows:

1. **Since $p_i > 0$ and $q_i > 0$, this means the pricing kernel ϕ_i is a positive function.**
2. **$E(\phi) = 1$.**

Normality of x_m: The CAPM

$$F_j = \sum_i (x_{j,i}\, q_i) = \sum_i p_i \; (\phi_i \; x_{j,i}) = E\,(\phi \; x_j)$$

$$F_j = E\,(x_j \phi) = E\,(\phi)\, E\,(x_j) + cov\,(\phi, x_j)$$

$$F_j = E\,(x_j) + cov\,(\phi, x_j)\,.$$

$$F_j = E\,(x_j) + cov\,[\phi(x_m), x_j]\,.$$

Covariance and joint probability

Covariance definition

$$cov(x, y) = E\{[x - E(x)][y - E(y)]\}.$$

$$\begin{aligned} E(x) &= p(x_1)x_1 + p(x_2)x_2 + ... \\ E(x) &= \sum_i p(x_i)x_i \end{aligned}$$

$$\begin{aligned} E(y) &= p(y_1)y_1 + p(y_2)y_2 + ... \\ E(y) &= \sum_i p(y_i)y_i \end{aligned}$$

Let $p(x_i, y_j)$ be the joint probability of x_i and y_j occurring, then

$$\begin{aligned} cov(x, y) &= p(x_1, y_1)[x_1 - E(x)][y_1 - E(y)] \\ &+ p(x_1, y_2)[x_1 - E(x)][y_2 - E(y)] \\ &+ ... \\ &+ p(x_i, y_j)[x_i - E(x)][y_j - E(y)] \end{aligned}$$

Stein's Lemma

If x, y are joint normally distributed

$$cov(g(x), y] = E[g'(x)]cov(x, y)$$

Hence

$$cov(x_j, \phi(x_m)] = E[\phi'(x_m)]cov(x_j, x_m)$$

Stein's Lemma

Examples:

1.

$$cov[g(x), y] = E[g'(x)]cov(x, y)$$

let

$$g(x) = a + bx$$

$$g'(x) = b$$

$$cov[g(x), y] = b\, cov(x, y)$$

2. let

$$g(x) = a + bx + cx^2$$

$$g'(x) = b + 2cx$$

$$cov[g(x), y] = [b + 2cE(x)]cov(x, y)$$

3. let

$$g(x) = e^x$$

$$g'(x) = e^x$$

$$cov[g(x), y] = E(e^x)cov(x, y)$$

CAPM: Derivation: Forward-Cash Flow Version

$$\begin{aligned} F_j &= E(x_j)\,E(\phi(x_m)) + cov(x_j, \phi(x_m)). \\ F_j &= E(x_j)\,E(\phi(x_m)) - \kappa cov(x_j, x_m). \\ &= E(x_j) - \kappa cov(x_j, x_m), \end{aligned}$$

$$\kappa = -E[\phi'(x_m)].$$

Since

$$F_m = E(x_m) - \kappa cov(x_m, x_m)$$

$$\kappa = \frac{E(x_m) - F_m}{var(x_m}$$

and

$$F_j = E(x_j) - \frac{E(x_m) - F_m}{var(x_m} cov(x_j, x_m)$$

CAPM: Applications

1. Cost of capital
   - For firm
   - For capital project

2. Valuation of non-listed equity

3. Mergers and acquisitions

For applications see Brealey and Myers, Copeland and Weston

APT Arbitrage Pricing Theory

We assume there are k factors and for factor f_k the factor loading is β_{jk}.

$$x_j = a_j + \sum_{k=1}^{K} \beta_{jk} f_k + \varepsilon_j$$

where ε_j is independent of f_k.

$$cov\,(x_j, \phi(x_m)) = \sum_{k=1}^{K} \beta_{jk} cov\,(f_k, \phi(x_m)) + cov(\varepsilon_j, \phi$$

1. $\varepsilon_j = 0$
2. $cov\,(\varepsilon_j, \phi(x_m)) = 0$

If $\varepsilon_j = 0$ or $cov\,(\varepsilon_j, \phi(x_m)) = 0$, then

$$\begin{aligned} cov\,(x_j, \phi(x_m)) &= \sum_{k=1}^{K} \beta_{jk} cov\,(f_k, \phi(x_m)) \\ F_j &= E\,(x_j) + \beta_{j1} cov\,(f_1, \phi(x_m)) \\ &+ \beta_{j2} cov\,(f_2, \phi(x_m)) + \cdots \end{aligned}$$

The Pricing Kernel:An Equilibrium Model

- **Assume a 'representative agent' economy**
- **Let $w_{t+T,i}$ be the wealth of the investor in state i at time $t+T$.**
- **Assume that the investor is endowed with investible wealth w_t at time t**
- **The investor can purchase state-contingent claims.**
- **Choose a set of state-contingent claims paying $w_{t+T,i}$, given a budget allocation of cash, w_t.**

We make the following additional assumptions:

1. **The investor maximises the expected value of a utility function $u(w_{t+T})$. Hence the investors problem is:**

   $$maxE\left[u\left(w_{t+T}\right)\right] = \sum_i p_i \ u(w_{t+T,i})$$

   subject to

   $$\sum_i w_{t+T,i} \ q_i \ B_{t,t+T} = w_t \qquad (10)$$

2. **The utility function has the properties $u'(w_{t+T}) > 0$ (non-satiation) and $u''(w_{t+T}) < 0$ (risk aversion).**

A two-state example:

$$\max p_1 u(w_{t+T,1}) + p_2 u(w_{t+T,2})$$

subject to

$$[w_{t+T,1} q_1 + w_{t+T,2} q_2] B_{t,t+T} = w_t$$

Lagrangian:

$$\begin{aligned} L &= p_1 u(w_{t+T,1}) + p_2 u(w_{t+T,2}) \\ &+ \lambda\{w_t B_{t,t+T}^{-1} - w_{t+T,1} q_1 - w_{t+T,2} q_2\} \end{aligned}$$

FOC for a maximum:

$$\frac{\partial L}{\partial w_{t+T,1}} = p_1 \; u'\left(w_{t+T,1}\right) - q_1 \lambda = 0.$$

$$\frac{\partial L}{\partial w_{t+T,2}} = p_2 \; u'\left(w_{t+T,2}\right) - q_2 \lambda = 0.$$

Summing over the states i

$$p_1\, u'(w_{t+T,1}) + p_2\, u'(w_{t+T,2}) = \lambda(q_1 + q_2)$$

or

$$E[u'(w_{t+T})] = \lambda,$$

since $q_1 + q_2 = 1.$

Substituting for λ

$$\frac{p_1\ u'\left(w_{t+T,1}\right)}{E\left[u'\left(w_{t+T}\right)\right]} = q_1,$$

$$\frac{p_2\ u'\left(w_{t+T,2}\right)}{E\left[u'\left(w_{t+T}\right)\right]} = q_2,$$

and hence the first order conditions are satisfied if

$$\frac{u'\left(w_{t+T,i}\right)}{E\left[u'\left(w_{t+T}\right)\right]} = \frac{q_i}{p_i} = \phi_i,\ \ \forall i.$$

$$\phi_i = \frac{u'\left(x_{m,i}\right)}{E\left[u'\left(x_{m,i}\right)\right]} :\ \forall i.$$

Hence we have

$$\phi = \phi(x_m),$$

Case 1: Risk Neutrality

$$u\left(w_{t+T}\right) = a + b\ w_{t+T},$$

$$u'\left(w_{t+T}\right) = b,$$

$$E[u'\left(w_{t+T}\right)] = b,$$

$$\phi_i\left(w_{t+T,i}\right) = \frac{u'\left(w_{t+T,i}\right)}{E\left[u'\left(w_{t+T}\right)\right]} = 1.$$

Forward Price Under Risk Neutrality:

In this case, the forward price is

$$\begin{aligned} F_{t,t+T} &= E\left(\phi\ x\right) \\ &= E\left(1 \cdot x\right) \\ &= E\left(F_{t+T,t+T}\right) \end{aligned}$$

Case 2: Utility is Quadratic

Assume utility is given by:

$$\begin{aligned} u(w_{t+T}) &= a + bw_{t+T} + \delta w_{t+T}^2 \\ u'(w_{t+T}) &= b + 2\delta w_{t+T} \\ \phi(w_{t+T}) &= \frac{u'(w_{t+T})}{E[u'(w_{t+T})]} \\ &= \frac{b + 2\delta w_{t+T}}{b + 2\delta E(w_{t+T})} \\ cov[\phi(w_{t+T}), x_{t+T}] &= \frac{2\delta cov(w_{t+T}, x_{t+T})}{b + 2\delta E(w_{t+T})} \end{aligned}$$

Then

$$F_{t,t+T} = E(x_{t+T}) + \kappa \; cov(w_{t+T}, x_{t+T}),$$

where

$$\kappa = \frac{2\delta}{b + 2\delta E(w_{t+T})}$$

Asset specific pricing kernel

For asset j, we can write

$$\begin{aligned} F_j &= E(\phi x_j) \\ &= E\left[E(\phi \mid x_j)\ x_j\right] \end{aligned}$$

$$F_j = E(\psi_j x_j)$$

$\psi_j > 0$. $E(\psi_j) = 1$.

Example:

Figure 1.2

$$\psi_{x_{1,g}} = E(\phi | x_1 = x_{1,g})$$

$$\psi_{x_{1,g}} = p_1\phi_1 + p_2\phi_2 + p_3\phi_3$$

Other Readings:

Huang C-F and R.H. Litzenberger (1988) Foundations for Financial Economics, North Holland, Chapters 1 and 4.

Pliska (1997) Introduction to Mathematical Finance, Blackwell, Chapters 1 and 2.

Cochrane (2001) Asset Pricing, Princeton, chs 1-6

Copeland and Weston (1988), Financial Theory and Corporate Policy, Addison-Wesley.

3 Option pricing and risk-neutral valuation

Options and Contingent Claims

A contingent claim on a cash flow x pays some function $g(x)$. The payoff $g(x)$ is contingent on the payoff x on the underlying asset.

A European-style call option maturing at time $t + T$, on a cash flow x pays

$$g(x) = max[x - k, 0]$$

A European-style put option maturing at time $t + T$, on a cash flow x pays

$$g(x) = max[k - x, 0]$$

The Normal Distribution

If $f(x)$ is normal with mean μ_x, and standard deviation σ_x:

$$f(x) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}[x-\mu_x]^2}$$

The standard normal distribution, has mean 0 and standard deviation 1 is

$$n(y) = \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}}$$

The cumulative normal distribution function is

$$F(x) = \int_{-\infty}^{x} f(u)du$$

The cumulative standard normal distribution function is

$$N(y) = \int_{-\infty}^{y} f(u)du$$

- **If $f(x)$ normal,**

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

-

$$E(x > a) = \int_{a}^{\infty} x f(x) dx$$

- **The normal distribution is the limit $n \rightarrow \infty$ of a binomial distribution with n steps**

The Joint Normal Distribution

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \cdot e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\frac{x-\mu_x}{\sigma_x}\frac{y-\mu_y}{\sigma_y}\right]}$$

- **If $f(x,y)$ joint normal, $f(x)$ and $f(y)$ are normal**
- **If $f(x,y)$ joint normal, the conditional distribution $f(x|y)$ is normal**
- **The regression of x on y is**

$$E(x|y) = \mu_x + \frac{\rho\sigma_x}{\sigma_y}(y-\mu_y)$$

A Simple Option Pricing Model

Assume quadratic utility for the representative investor *and* **joint-normal distribution**

From ch 1

$$\begin{aligned} u(x_m) &= a + bx_m + \delta x_m^2 \\ \phi(x_m) &= \frac{u'(x_m)}{E[u'(x_m)]} \\ &= \frac{b + 2\delta x_m}{b + 2\delta E(x_m)} \end{aligned}$$

Hence in this case

$\phi = A + Bx_m.$

$$F[g(x)] = E[g(x)] + cov(g(x), A + Bx_m)$$

Using Stein's lemma

$$F[g(x)] = E[g(x)] + BE[g'(x)]\, cov(x, x_m)$$

The contingent claim is a call option where $g(x) = \max(x - k, 0)$.

$E[g'(x)] = \text{prob}(x > k)$ and hence

$$F[g(x)] = E[g(x)] + B[\text{prob}(x > k)]\, cov(x, x_m) \tag{11}$$

$$F_m = E[x_m] + B\ var(x_m)$$

$$B = \frac{F_m - E(x_m)}{var(x_m)}$$

An application: Valuation of Corporate debt

Assumptions:

1. **Company assets produces cash flow x_j at time $t+T$**
2. **x_j and x_m are joint normal**
3. **Bonds outstanding B_{t+T} to be repaid at time $t+T$**
4. **Equity is call option receives:**

$$g(x) = max[x_j - B_{t+T}, 0]$$

at $t+T$

5. Forward value of equity is

$$F[g(x)] = E[g(x)] + B\,[\text{prob}(x > k)]cov(x_j, x_m)$$

6. Spot value of equity is

$$S_t = F[g(x)]B_{t,t+T}$$

7. Value of company assets is V_t

8. Value of the debt is

$$V_t - S_t$$

The Black-Scholes Model

- **Purpose**
  - **To price options on stocks**
  - **To value corporate liabilities**
  - **To evaluate credit risk**
- **Assumptions**
  - **Lognormal asset price**
  - **No dividends**
  - **Option is European-style**
- **Main Features**
  - **Preference-free relationship**
  - **Mean of asset not required**
  - **Inputs: asset price, volatility**
- **Generalisations**
  - **Forex options**
  - **American-style options**

Pricing of Contingent Claims

$$F_{t,t+T}\left[g\left(x_{j,t+T}\right)\right] = \sum_i q_i g\left(x_{j,t+T,i}\right)$$

$$= \sum_i p_i \phi_i g\left(x_{j,t+T,i}\right)$$

$$= E\left[g\left(x_{j,t+T}\right)\phi\right]$$

$$= E\left[g(x_j)E(\phi|x_j)\right]$$

$$= E\left[g(x_j)\psi(x_j)\right]$$

Characteristics of the Pricing Kernel, ψ_j

Assume that

$$\psi_j = \alpha x_j^{\beta}$$

where α and β are constants.

The asset specific pricing kernel has constant elasticity.

The elasticity of the pricing kernel defined by the relationship:

$$\eta = -\frac{\partial \psi_j / \psi_j}{\partial x_j / x_j}$$

Special Case of Options on x_m

Conditions for constant elasticity

- In this case $\psi_j = \phi(x_m)$
- $\phi(x_m)$ has constant elasticity if the representative investor has power utility
- If $u(x_m) = x_m^\gamma$ then ϕ has constant elasticity

The Lognormal Distribution

If $\ln(x)$ is normal with

$$E[\ln(x)] = \mu_x, var[\ln(x)] = \sigma_x^2$$

then x is lognormally distributed

If $f(\ln x)$ is normal with mean μ_x, and standard deviation σ_x:

$$f(\ln x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}[\ln x - \mu_x]^2}$$

- **The lognormal distribution is the limit of a process where $\ln x$ is binomial**
- **Assume a stock price S_t goes up by a proportion u with probability p or down by d with probability $(1 - p)$ then S_t is log-binomial**

Further Properties of Lognormal Variables

1. **The expected value of a lognormal variable: Let Y be lognormal, with $y = \ln(Y)$**

$$E(Y) = E(e^y) = e^{\mu_y + \frac{1}{2}\sigma_y^2}.$$

2. **Also,**

$$E(Y^b) = E(e^{by}) = e^{b\mu_y + \frac{1}{2}b^2\sigma_y^2}.$$

3. **If x, y are lognormal, xy is lognormal.**

$$\ln xy = \ln x + \ln y$$

It follows

$$E(\ln xy) = E(\ln x) + E(\ln y)$$

or

$$E(\ln xy) = \mu_x + \mu_y$$

$$\begin{aligned} var(\ln xy) &= var(\ln x) + var(\ln y) \\ &+ 2cov(\ln x, \ln y) \end{aligned}$$

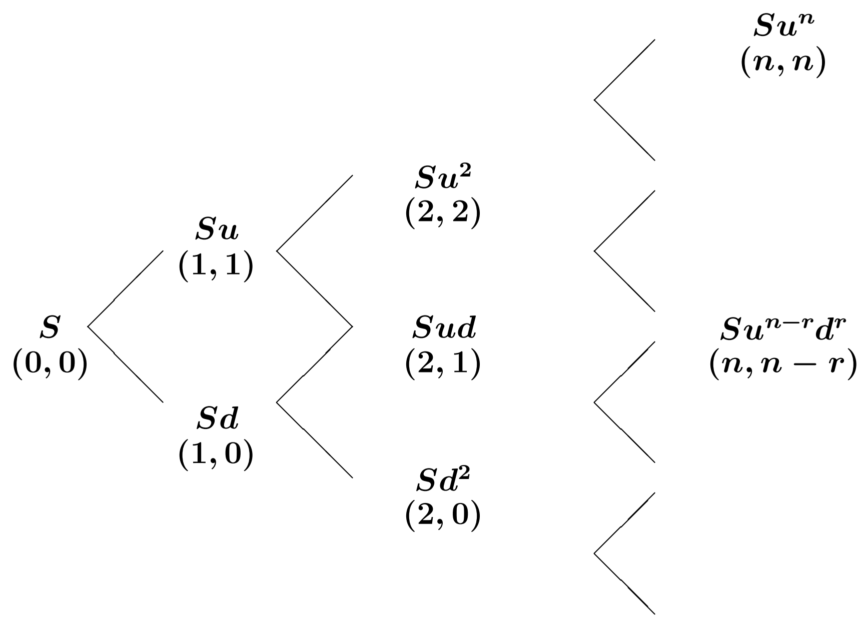
or

$$var(\ln xy) = \sigma_x^2 + \sigma_y^2 + 2\sigma_{x,y}$$

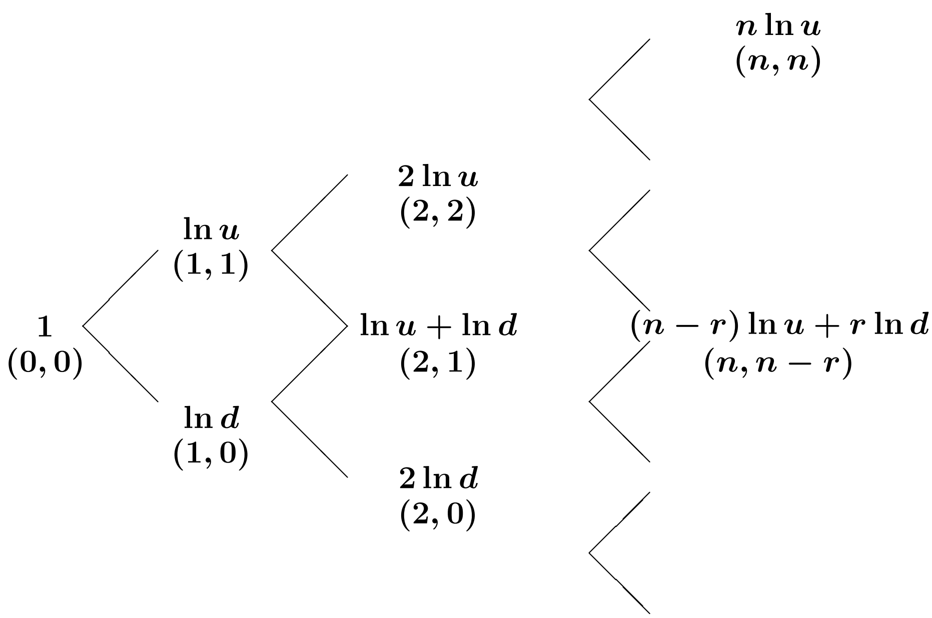
4. It follows:

$$\begin{aligned} E(xy) &= e^{E(\ln xy) + \frac{1}{2} var(\ln xy)} \\ &= e^{\mu_x + \mu_y + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_y^2 + \sigma_{x,y}} \end{aligned}$$

A General Binomial Process



A Log-Binomial Process



Lognormal x_j and the Black-Scholes Model

Properties of Lognormality

Pricing Kernel **Risk-adjusted PDF**

Forward Price of x_j **RNVR**

Black-Scholes Model

Module 1: Notation and Lognormal Properties

$$E[\ln x] = \mu_x$$

$$var[\ln x] = \sigma_x^2$$

$$f(\ln x) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}[\ln x - \mu_x]^2}$$

$$E(x_j) = e^{\mu_x + \frac{1}{2}\sigma_x^2}$$

$$E(x_j^\beta) = e^{\beta\mu_x + \frac{1}{2}\beta^2\sigma_x^2}$$

Option payoff:

$$g(x_j) = g\left(e^{\ln x_j}\right) = h(\ln x_j)$$

Module 2: Asset Specific Pricing Kernel

$$\begin{aligned} F\left[g\left(x_j\right)\right] &= E\left[g\left(x_j\right)\phi(x_m)\right] \\ &= E\left[g\left(x_j\right)E\left[\phi(x_m)|x_j\right]\right] \\ &= E\left[g\left(x_j\right)\psi(x_j)\right] \end{aligned}$$

Assume that

$$\psi_j = \alpha x_j^{\beta}$$

where α and β are constants.

Elasticity:

$$\eta = -\frac{\partial\psi_j/\psi_j}{\partial x_j/x_j}$$

$$\eta = -\alpha\beta x_j^{\beta-1}\frac{x_j}{\alpha x_j^{\beta}} = -\beta$$

$E(\psi_j) = 1$ implies that

$$\alpha = e^{-\beta\mu_x - \frac{1}{2}\beta^2\sigma_x^2}.$$

Module 3: Risk-Adjusted PDF

PDF of x_j:

$$f(\ln x_j) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}[\ln x_j - \mu_x]^2}$$

Risk-adjusted PDF of x_j:

$$\hat{f}(\ln x_j) = f(\ln x)\psi(x_j) = \alpha x_j^{\beta} \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}[\ln x_j - \mu_x]}$$

$$\hat{f}(\ln x_j) = e^{-\beta\mu_x - \frac{1}{2}\beta^2\sigma_x^2} x_j^{\beta} \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}[\ln x_j - \mu_x]^2}$$

$$\hat{f}(\ln x_j) = \frac{1}{\sigma_x \sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}[\ln x_j - (\mu_x + \beta\sigma_x^2)]^2}$$

Module 4: Forward Asset Price of x_j

From ch 1:

$$F_j = E[x_j \psi(x_j)]$$

$$F_j = e^{\mu_x + \mu_\psi + \frac{1}{2}\sigma_x^2 + \frac{1}{2}\sigma_\psi^2 + \sigma_{x\psi}}$$

since $E(\psi) = 1$

$$F_j = e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x\psi}}$$

$$\ln F_j - \frac{1}{2}\sigma_x^2 = \mu_x + \sigma_{x\psi}$$

Module 5: The Lognormal RNVR

Under Risk Neutrality:

$$E\left[g\left(x_j\right)\right] = E\left[h\left(\ln x_j\right)\right]$$

$$= \int h(\ln x_j) f(\ln x_j) d \ln x_j$$

$$F_j = E(x_j) = e^{\mu_x + \frac{1}{2}\sigma_x^2}$$

$$E[g(x_j)] = \int h(\ln x_j) \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}[\ln x_j - (\ln F - \frac{1}{2}\sigma_x^2)]^2} d \ln$$

Under Risk Aversion

$$F[g(x_j)] = \int h(\ln x_j) \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}[\ln x_j - (\ln F - \frac{1}{2}\sigma_x^2)]^2} d \ln$$

Module 6: Option Prices: The Black-Scholes Model

$$F[g(x_j)] = E\left[g\left(x_j\right)\psi\left(x_j\right)\right]$$

$$F[g(x_j)] = \int h(\ln x_j)\frac{1}{\sigma_x\sqrt{2\pi}}e^{-\frac{1}{2\sigma_x^2}[\ln x_j - (\ln F - \frac{1}{2}\sigma_x^2)]^2}d\ln$$

The Black-Scholes Model: A Call Option

A European-style call option on x_j, with strike price k has a payoff at time $t+T$:

$$g(x_j) = max(x_j - k, 0).$$

or

$$h(\ln x_j) = max(e^{\ln x_j} - e^{\ln k}, 0).$$

$$F\left[g\left(x\right)\right] = \int_{-\infty}^{\infty} max\left(e^{\ln x_j} - k, 0\right) \hat{f}\left(\ln x_j\right) d\ln x_j$$

where

$$\hat{f}\left(\ln x_j\right) = \frac{1}{\sigma_x\sqrt{2\pi}} e^{-\frac{1}{2\sigma_x^2}\left[\ln x_j - (\ln F_j - \frac{1}{2}\sigma_x^2)\right]^2}$$

$$F\left[g\left(x\right)\right] = \int_{\ln k}^{\infty} \left(e^{\ln x_j} - k\right) \hat{f}\left(\ln x_j\right) d\ln x_j$$

$$\begin{aligned} F\left[g\left(x\right)\right] \; &= \; \int_{\ln k}^{\infty} e^{\ln x_j} \hat{f}\left(\ln x_j\right) d\ln x_j \\ &- \; k \int_{\ln k}^{\infty} \hat{f}\left(\ln x_j\right) d\ln x_j . \end{aligned}$$

The Normal Distribution

Probability $y > a$

and

$$\int_a^\infty f(y)dy.$$

$$\int_a^\infty f(y)dy = 1 - N\left(\frac{a - \mu_y}{\sigma_y}\right) = N\left(\frac{\mu_y - a}{\sigma_y}\right).$$

The Normal Distribution

Expected value of e^y given $y > a$

$$\int_a^\infty e^y f(y)dy$$

$$\int_a^\infty e^y f(y)dy = N\left(\frac{\mu_y - a}{\sigma_y} + \sigma_y\right) e^{\mu_y + \frac{1}{2}\sigma_y^2}.$$

Evaluate the integrals

$y = \ln x_j$ **and** $a = \ln k$.

The Normal Distribution

Applying the above results and substituting the mean of $\hat{f}(\ln x_j)$, $\mu_x = \ln(F_j) - \frac{\sigma_x^2}{2}$ and $a = \ln k$, we have

$$N\left(\frac{\mu_y - a}{\sigma_y}\right) = N\left(\frac{\ln F_j - \frac{\sigma_x^2}{2} - \ln k}{\sigma_x}\right)$$

and

$$N\left(\frac{\mu_y - a}{\sigma_y} + \sigma_y\right) e^{\mu_y + \frac{1}{2}\sigma_y^2}$$
$$= F_j N\left(\frac{\ln F_j - \frac{\sigma_x^2}{2} - \ln k + \sigma_x^2}{\sigma_x}\right)$$

The Black-Scholes Model: Forward Price

$$F\left[g\left(x\right)\right] = F_j N\left(\frac{\ln\frac{F}{k} - \frac{\sigma_x^2}{2} + \sigma_x^2}{\sigma_x}\right) - kN\left(\frac{\ln\frac{F}{k} - \frac{\sigma_x^2}{2}}{\sigma_x}\right).$$

Asset forward price F_j

The (logarithmic) variance σ_x^2,

The strike price of the option, k.

The Black-Scholes Model: Spot Price

$$S_t[g(x_j)] = B_{t,t+T} F_j N\left(\frac{\ln\frac{F}{k} - \frac{\sigma_x^2}{2} + \sigma_x^2}{\sigma_x}\right) - B_{t,t+T} k N\left(\frac{\ln\frac{F}{k} - \frac{\sigma_x^2}{2}}{\sigma_x}\right).$$

or, using conventional notation:

$$S_t[g(x_j)] = B_{t,t+T} F_j N(d_1) - B_{t,t+T} k N(d_2),$$

where

$$d_1 = \frac{\ln\frac{F_j}{k} + \frac{\sigma_x^2}{2}}{\sigma_x}$$
$$d_2 = d_1 - \sigma_x.$$

The Black-Scholes Model: Applications

1. **Non-dividend paying assets:**
2. **Assets paying a non-stochastic dividend:**
3. **Assets paying a stochastic proportional dividend**

The Black-Scholes Model: Non-dividend paying assets

In this case, spot-forward parity for the underling asset means that the spot price of the asset is

$$S_t = F_j B_{t,t+T},$$

where $B_{t,t+T} = e^{-rT}$ **and** r **is the continuously compounded interest rate.**

$$S_t[g(x_j)] = S_t N\left(d_1\right) - ke^{-rT} N\left(d_2\right), \tag{12}$$

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_t}{k} + rT + \frac{\sigma_x^2}{2}}{\sigma_x} \\ d_2 &= d_1 - \sigma_x \end{aligned}$$

The Black-Scholes Model: Assets paying a non-stochastic dividend

known dividend D_{t+T} at time $t+T$.

In this case, spot-forward parity implies

$$S_t = (F_j + D_{t+T})B_{t,t+T},$$

$$S_t[g(x_j)] = (S_t - D_{t+T}e^{-rT})N\left(d_1\right) - ke^{-rT}N\left(d_2\right),$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_t - D_{t+T}e^{-rT}}{k} + \frac{\sigma_x^2}{2} + rT}{\sigma_x} \\ d_2 &= d_1 - \sigma_x. \end{aligned}$$

The Black-Scholes Model: Assets paying a stochastic proportional dividend

$$D_{t+T} = \delta x_j$$

$$S_t = F_j(1+\delta)B_{t,t+T}.$$

$$S_t[g(x_j)] = \frac{S_t}{(1+\delta)}N\left(d_1\right) - ke^{-rT}N\left(d_2\right).$$

where

$$\begin{aligned} d_1 &= \frac{\ln\frac{S_t}{(1+\delta)} + rT - \ln k + \frac{\sigma_x^2}{2}}{\sigma_x} \\ d_2 &= d_1 - \sigma_x. \end{aligned}$$

Spot-Forward Parity

1. **Non-dividend paying assets:**

$$S_t = F_j B_{t,t+T},$$

$$F_j = \frac{S_t}{B_{t,t+T}}$$

2. **Assets paying a non-stochastic dividend:**

$$S_t = (F_j + D_{t,t+T})B_{t,t+T},$$

$$F_j = \frac{S_t}{B_{t,t+T}} - D_{t+T}$$

3. **Assets paying a stochastic proportional dividend**

$$S_t = F_j(1+\delta)B_{t,t+T}.$$

$$F_j = \frac{S_t}{B_{t,t+T}}\left(\frac{1}{1+\delta}\right)$$

The Black-Scholes Model: Extensions

1. **Multivariate Options:**
2. **Compound Options:**
3. **Bermudan-style options**
4. **American-style options**
5. **Non-Constant Elasticity**
6. **Non-Lognormal Asset Distributions**

4 Multi-period asset pricing

Asset prices in a Multi-period Economy

1. **Chapter 5: Extension of ch 1 model to multiple periods**
2. **Chapter 6: Application to analysis of Forward contracts and Futures contracts**

Motivation

1. **Valuation of a company: more realistic model**
2. **Capital budgeting: Should an Investment project be accepted**
3. **Does the CAPM extend and apply in a multi-period world?**
4. **Basic problem**
   - **Multiple dates**
   - **Multiple states at each date**

Asset prices in a Multi-period Economy: Introduction

- **Valuation under certainty**
- **Time-state preference approach**
- **Rational expectations approach**

Asset prices in a Multi-period Economy: Certainty Case

$$\begin{array}{cccc} t+1 & t+2 & ... & t+n \\ X_{j,t+1} & X_{j,t+2} & ... & X_{j,t+n} \\ B_{t,t+1} & B_{t,t+2} & ... & B_{t,t+n} \end{array}$$

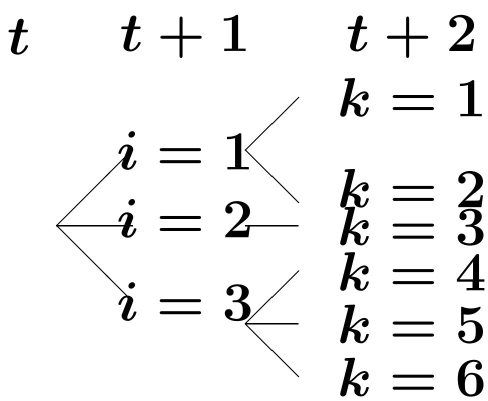
$$\begin{aligned} S_{j,t} &= B_{t,t+1}(X_{j,t+1} + S_{j,t+1}) \\ S_{j,t+1} &= B_{t+1,t+2}(X_{j,t+2} + S_{j,t+2}) \\ S_{j,t+2} &= B_{t+2,t+3}(X_{j,t+3} + S_{j,t+3}) \\ ... &= ... \\ S_{j,t+n-1} &= B_{t+n-1,t+n}X_{j,t+n} \end{aligned}$$

$$\begin{aligned} S_{j,t} &= B_{t,t+1}X_{j,t+1} + B_{t,t+1}B_{t+1,t+2}X_{j,t+2}... \\ &+ B_{t,t+1}B_{t+1,t+2}...B_{t+n-1,t+n}X_{j,t+n} \end{aligned}$$

Asset prices in a Multi-period Economy: Basic Set-up

1. **Markets are complete. Investors can purchase claims that pay \$1 if and only if a given state occurs at a given point in time.**
2. **Assume there are just two periods; period $t–t+1$ and period $t+1–t+2$.**
3. **Value a stock j which pays dividends $x_{j,t+1}$ at time $t+1$ and $x_{j,t+2}$ at time $t+2$.**
4. **There are $i = 1, 2, \cdots, I$ states at time $t+1$ and there are $k = 1, 2, \cdots, K$ states at time $t+2$.**

Asset prices in a Multi-period Economy: Basic Set-up



Time-State Preference Approach

1. **Treat each period cash flow as separate valuation**
2. **Define** $\phi_{t,t+1} = q_i/p_i$
3. **Define** $\phi_{t,t+2} = q_k/p_k$

   and so on.

   Then applying ch 1 for each T**:**

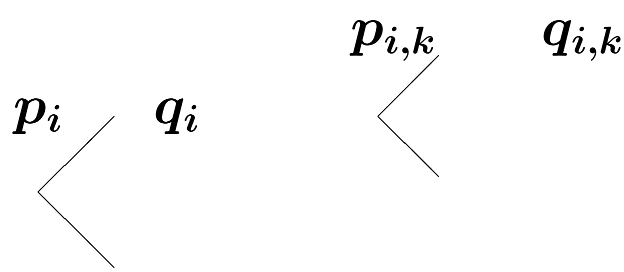
$$S_t = \sum_{T=1}^{n} F_{t,t+T} B_{t,t+T},$$

   where

$$F_{t,t+T} = E\left[\phi_{t,t+T} x_{t+T}\right].$$

Rational Expectations Approach: Basic Set-up

Consider the following state diagram:-



p_i is the probability of state i occuring at time $t+1$

$p_{i,k}$ is the conditional probability of state k occuring at $t+2$

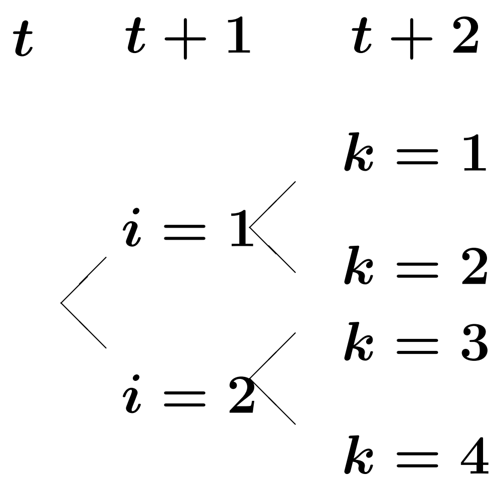
q_i is the forward price of a dollar paid *if and only if* state i occurs.

$q_{i,k}$ is the (conditional) forward price, in state i of a dollar paid *if and only if* state k occurs.

Equivalence of TSP and RE Prices: Example

Let $i = 1, 2$, $k = 1, 2, 3, 4$

Assume a state space as follows:



Equivalence of TSP and RE Prices: Example

1. Zero-coupon bond prices

$$\begin{aligned} B_{t,t+1} &= 0.9 \\ B_{t+1,t+2,1} &= 0.9 \\ B_{t+1,t+2,2} &= 0.8 \end{aligned}$$

2. Forward state prices

$$\begin{aligned} q_i,\ i=1:\ q_1 &= 4/9 \\ q_i,\ i=2:\ q_2 &= 5/9 \\ q_{i,k},\ i=1,k=1:\ q_{1,1} &= 4/9 \\ q_{i,k},\ i=1,k=2:\ q_{1,2} &= 5/9 \\ q_{i,k},\ i=2,k=3:\ q_{2,3} &= 1/2 \\ q_{i,k},\ i=2,k=4:\ q_{2,4} &= 1/2 \end{aligned}$$

Equivalence of TSP and RE Prices: Example

t	$t+1$	$t+2$
?		1
(4/9)(0.9).(4/9)(0.9)	0.4	
	−0.4	1

Note: 0.4 is spot price at date 1, in state 1

Strategy ensures $1 in state 1 at date 2

Rational Expectations Approach

Using ch 1 method:

$$\begin{aligned} S_t &= B_{t,t+1} \sum_i p_i \phi_{t,t+1,i} \; S_{t+1,i} \\ &= B_{t,t+1} \; E_t \left(\phi_{t,t+1} S_{t+1} \right) \end{aligned}$$

This is the value of x_{t+2} at time t.

Substituting for S_{t+1}:

$$S_t = B_{t,t+1} E_t \left[\phi_{t,t+1} B_{t+1,t+2} \; E_{t+1} \left(\phi_{t+1,t+2} \; x_{t+2} \right) \right]$$

The Fundamental Equation of Valuation

The value of any cash flow $x_{j,t+T}$:

$$\begin{aligned} S_{j,t} &= B_{t,t+1} E_t[\phi_{t,t+1} B_{t+1,t+2} E_{t+1}(\phi_{t+1,t+2} \\ &\quad .... \; B_{t+T-1,t+T} E_{t+T-1}(\phi_{t+T-1,t+T} x_{j,t+T}))] \end{aligned}$$

where $\phi_{\tau,\tau+1}$ is the period-by-period pricing kernel.

The Relationship Between the Pricing Kernels when Interest Rates are Non-stochastic

The cost of a claim on state k is $B_{t,t+2}q_k$.

Alternatively, cost is

$B_{t,t+1}q_iB_{t+1,t+2}q_{i,k}$.

Hence

$$B_{t,t+2}q_k = B_{t,t+1}q_iB_{t+1,t+2}q_{i,k}.$$

$$B_{t,t+2}\frac{q_k}{p_k} = B_{t,t+1}\frac{q_i}{p_i}B_{t+1,t+2}\frac{q_{i,k}}{p_{i,k}},$$

and hence

$B_{t,t+2}\phi_{t,t+2} = B_{t,t+1}\phi_{t,t+1}B_{t+1,t+2}\phi_{t+1,t+2}$.

But since in this case, $B_{t,t+2} = B_{t,t+1}B_{t+1,t+2}$, we have

$$\phi_{t,t+2} = \phi_{t,t+1}\ \phi_{t+1,t+2}.$$

or, in general

$$\phi_{t,t+T} = \phi_{t,t+1}\ \phi_{t+1,t+2}....\phi_{t+T-1,t+T}.$$

Rational Expectations Approach: Basic Set-up

We now define the period-by-period pricing kernel by the relationship:

$$\phi_{t+1,t+2,i,k} = \frac{q_{i,k}}{p_{i,k}}.$$

Using this we can write:

$$\begin{aligned} S_{t+1,i} &= B_{t+1,t+2,i} \sum_k p_{i,k} \phi_{t+1,t+2,i,k} \, x_{t+2,k} \\ &= B_{t+1,t+2,i} \, E_{t+1,i} \left(\phi_{t+1,t+2,i} \, x_{t+2} \right) \end{aligned}$$

This is the value of x_{t+2} at time $t+1$ in state i.

The Structure of State-Contingent Claim Prices

	Time t	Time $t+1$	Time t
Spot price at t of \$1 paid at $t+2$	q_k^*		
Spot price at t of \$1 paid at $t+1$	q_i^*		
Spot price at $t+1$ of \$1 paid at $t+2$		$q_{i,k}^*$	
Period − by − period valuation	$q_i^* q_{i,k}^*$	$q_{i,k}^*$	\$1
One − long − period valuation	q_k^*		\$1

$$
\begin{aligned}
q_i^* &= q_i B_{t,t+1} \\
q_k^* &= q_k B_{t,t+2} \\
q_{i,k}^* &= q_{i,k} B_{t+1,t+2,i}
\end{aligned}
$$

Equivalence of TSP and RE Prices

t	$t+1$	$t+2$
$-q_k B_{t,t+2}$		1
$-q_i B_{t,t+1} q_{i,k} B_{t+1,t+2,i}$	$+q_{i,k} B_{t+1,t+2,i}$	
	$-q_{i,k} B_{t+1,t+2,i}$	1

t	$t+1$	$t+2$
$-q_k^*$		1
$-q_i^* q_{i,k}^*$	$+q_{i,k}^*$	
	$-q_{i,k}^*$	1

Rational Expectations Equilibrium: Joint-Normal Cash Flows

$$S_{j,t} = B_{t,t+1}E_t[\phi_{t,t+1}B_{t+1,t+2}E_{t+1}(\phi_{t+1,t+2}x_{j,t+2})]$$

Alternatively, this pricing equation can be written as

$$S_{j,t} = B_{t,t+1}E_t[\phi_{t,t+1}S_{j,t+1}],$$

where

$$S_{j,t+1,i} = B_{t+1,t+2,i}E_{t+1,i}(\phi_{t+1,t+2,i}x_{j,t+2})].$$

From ch 1, the CAPM holds over $t+1$ to $t+2$:

$$\begin{aligned} S_{j,t+1,i} &= B_{t+1,t+2,i}[E_{t+1,i}(x_{j,t+2}) \\ &- \lambda_{t+1,i}cov_{t+1,i}(x_{j,t+2}, x_{m,t+2})]. \end{aligned}$$

where $\lambda_{t+1,i}$ is the market price of risk in state i at $t+1$

Multi-period Valuation: Joint-Normal Cash Flows

$$\begin{aligned} S_{j,t+1,i} &= B_{t+1,t+2,i}[E_{t+1,i}(x_{j,t+2}) \\ &- \lambda_{t+1,i} cov_{t+1,i}(x_{j,t+2}, x_{m,t+2})]. \end{aligned}$$

Now assume:

1. $B_{t+1,t+2,i}$ **is non-stochastic**
2. $\lambda_{t+1,i}$ **is non-stochastic**

These assumptions were made by Stapleton and Subrahmanyam *Econometrica* **(1978)**

Then it follows that:

The value, $S_{j,t+1,i}$**, is normally distributed.**

Also $S_{m,t+1,i}$**, is normally distributed and the conditions for the period-by-period CAPM hold.**

Multi-period Valuation: Joint-Normal Cash Flows

$$S_{j,t} = B_{t,t+1}\left[E_t\left(S_{j,t+1}\right) - \lambda_t cov\left(S_{j,t+1}, S_{m,t+1}\right)\right].$$

Substituting for the time $t+1$ prices, we can write:

$$\begin{aligned} S_{j,t} &= B_{t,t+1}B_{t+1,t+2}E_t(x_{j,t+2}) \\ &- B_{t,t+2}\lambda_t cov_t\left[E_{t+1}(x_{j,t+2}), E_{t+1}(x_{m,t+2})\right] \\ &- B_{t,t+2}\lambda_{t+1} cov_{t+1}\left[x_{j,t+2}, x_{m,t+2}\right] \end{aligned}$$

Multi-Period Valuation: Extensions

1. If $B_{t+1,t+2,i}$ is stochastic: Intertemporal CAPM (Merton, Long)
2. Equilibrium model to determine $\phi_{t,t+1}$, ch 5.8
3. Consumption CAPM, see Cochrane
4. RE approach, see Pliska
5. Applications: Futures v Forward prices

5 Forward and futures prices

Forward Contracts and Futures Contracts: Motivation

- **Futures contracts are the way most contracts are traded on public exchanges**
- **In the case of interest-rate related securities the difference between forward and futures prices is important**
- **Determine futures prices of options (as traded on LIFFE)**
- **Futures prices are expected values (under risk-neutral measure)**

Forward Contracts and Futures Contracts

- **A** *forward contract* **is an agreement made at a point in time t to purchase or sell an asset at a later date $t + T$.**
- **A** *futures contract* **is similar to a forward contract but is marked to market on a daily basis as time progresses from t to $t + T$.**

Futures Contracts: Payoffs

A long futures contract made at time t, with maturity T, to buy an asset at a price $H_{t,t+T}$ has a payoff $[H_{t+\tau,t+T} - H_{t+\tau-1,t+T}]$ at time $t+\tau$.

Hence the payoff at time $t+1$ is $H_{t+1,t+T} - H_{t,t+T}$

A short futures contract made at time t, with maturity T, to sell an asset at a price $H_{t,t+T}$ has a payoff $[-H_{t+\tau,t+T} + H_{t+\tau-1,t+T}]$ at time $t+\tau$.

Hence the payoff at time $t+1$ is $-H_{t+1,t+T} + H_{t,t+T}$

Forward Contracts: Payoffs

The payoff on a long forward contract is $[S_{t+T} - F_{t,t+T}]$ at time $t + T$,

The payoff on a short forward contract is $[-S_{t+T} + F_{t,t+T}]$ at time $t + T$

Forward and Futures: Payoffs

t	$t+1$	$t+2$	$t+3$	$\cdots$	$t+T$
Futures contract	$H_{t+1,t+T}$ $-H_{t,t+T}$	$H_{t+2,t+T}$ $-H_{t+1,t+T}$	$H_{t+3,t+T}$ $-H_{t+2,t+T}$	$\cdots$ $\cdots$	$H_{t+T,t+T}$ $-H_{t+T-1,t+T}$
Forward contract					S_{t+T} $-F_{t,t+T}$

Long Futures and Long Forward

- **Futures is marked-to-market daily**
- **Forward pays off at end of contract**

Characterization of Futures Price

[CIR (1981) Proposition 1]

Consider an asset with a price $\widetilde{S}_{t+T}$ at time $t+T$. The futures price of the asset, $H_{t,t+T}$, is the time t spot price of an asset which has a payoff

$$\frac{\widetilde{S}_{t+T}}{B_{t,t+1}\widetilde{B}_{t+1,t+2}\cdots\widetilde{B}_{t+T-1,t+T}}$$

at time $t+T$.

Cox J.C., J. E. Ingersoll Jr and S. A. Ross (1981) The relationship between forward prices and futures prices, *Journal of Financial Economics*, 9, pp.321-346.

A Three-year Example

Strategy

- Time 0: invest $H_{0,3}$ overnight
- Time 0: $\frac{1}{B_{0,1}}$ long futures
- Time 1: re-invest proceeds overnight
- Time 1: $\frac{1}{B_{0,1}B_{1,2}}$ long futures
- Time 2: re-invest proceeds overnight
- Time 2: $\frac{1}{B_{0,1}B_{1,2}B_{2,3}}$ long futures

Characterization of Futures Price

Time	Profits from futures	Gain from investment	Net position
0	–	–	$H_{0,3}$
1	$\frac{1}{B_{0,1}}[H_{1,3} - H_{0,3}]$	$\frac{1}{B_{0,1}}[H_{0,3}]$	$\frac{1}{B_{0,1}}[H_{1,3}]$
2	$\frac{1}{B_{0,1}B_{1,2}}[H_{2,3} - H_{1,3}]$	$\frac{1}{B_{0,1}B_{1,2}}[H_{1,3}]$	$\frac{1}{B_{0,1}B_{1,2}}[H_{2,3}]$
3	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}}[H_{3,3} - H_{2,3}]$	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}}[H_{2,3}]$	$\frac{1}{B_{0,1}B_{1,2}B_{2,3}}[H_{3,3}]$

Characterization of Futures Price

- The futures price for immediate delivery at $\boldsymbol{T = 3}$ is $\boldsymbol{H_{3,3} = S_3}$
- The strategy turns an investment of $\boldsymbol{H_{0,3}}$ into a cash flow of $\boldsymbol{\frac{S_3}{B_{0,1}B_{1,2}B_{2,3}}}$.
- In general, an investment of $\boldsymbol{H_{t,t+T}}$ can be turned into $\boldsymbol{\frac{S_{t+T}}{B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}\ldots B_{t+T-1,t+T}}}$.
- Hence $\boldsymbol{H_{t,t+T}}$ must be the value of this payoff.

Characterization of Forward Price

[CIR proposition 2]

Consider an asset with a price $\widetilde{\boldsymbol{S}}_{\boldsymbol{t+T}}$ at time $\boldsymbol{t+T}$. The forward price of the asset, $\boldsymbol{F_{t,t+T}}$, is the time $\boldsymbol{t}$ spot price of an asset which has a payoff

$$\frac{\widetilde{\boldsymbol{S}}_{\boldsymbol{t+T}}}{\boldsymbol{B_{t,t+T}}}$$

at time $\boldsymbol{t+T}$.

Characterization of Forward Price

- Invest $F_{t,t+T}$ in a T-maturity risk-free bond at time t at the long bond price, $B_{t,t+T}$.
- Take out $1/B_{t,t+T}$ long forward contracts to buy the asset.
- Time $t+T$, the payoff of the risk-free bond investment is $F_{t,t+T}/B_{t,t+T}$,
- Forward contract payoff is $\left(S_{t+T} - F_{t,t+T}\right)/B_{t,t+T}$.
- At time $t+T$, the combined position is $S_{t+T}/B_{t,t+T}$.
- So $F_{t,t+T}$ is the time t value of $\widetilde{S}_{t+T}/B_{t,t+T}$.

Pricing under Rational Expectations

S_t is the value of x_{t+3}:

$$S_t = B_{t,t+1}E_t\{\phi_{t,t+1}B_{t+1,t+2}E_{t+1}[\phi_{t+1,t+2} B_{t+2,t+3}E_{t+2}(\phi_{t+2,t+3}x_{t+3})]\},$$

where $\phi_{\tau,\tau+1}$, are the period-to-period pricing kernels.

$$S_t = E_t\{\phi_{t,t+1}E_{t+1}[\phi_{t+1,t+2}E_{t+2}(\phi_{t+2,t+3} B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}x_{t+3})]\},$$

The futures price $H_{t,t+3}$ is the time-t value of $\frac{x_{t+3}}{B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}}$

$$S_t = E_t\{\phi_{t,t+1}E_{t+1}[\phi_{t+1,t+2}E_{t+2}(\phi_{t+2,t+3} B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}\frac{x_{t+3}}{B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}})]\}$$

$$H_{t,t+3} = E_t\{\phi_{t,t+1}E_{t+1}[\phi_{t+1,t+2}E_{t+2}(\phi_{t+2,t+3}x_{t+3})]\}$$

A Great Result

$$H_{t,t+3} = E_t\{\phi_{t,t+1}E_{t+1}[\phi_{t+1,t+2}E_{t+2}(\phi_{t+2,t+3}x_{t+3})]\}$$

or,

$$\begin{aligned} H_{t,t+3} &= E_t^Q\{E_{t+1}^Q[E_{t+2}^Q(x_{t+3})]\} \\ &= E_t^Q(x_{t+3}) \end{aligned}$$

Futures Price: Interpretation

- Under risk neutrality, futures = expected payoff
- Futures price has 'martingale property' under $\boldsymbol{Q}$ (risk-neutral) measure
- Compounding effect offsets stochastic discounting
- Under risk neutrality, forward $\neq$ expected payoff

The Forward Price

the forward price of x_{t+3} is the spot price of an asset paying $x_{t+3}/B_{t,t+3}$ at time $t+3$.

$$F_{t,t+3} = E_t^Q\left(\frac{B_{t,t+1}B_{t+1,t+2}B_{t+2,t+3}x_{t+3}}{B_{t,t+3}}\right).$$

$$F_{t,t+T} = E_t^Q\left[\prod_{\tau=1}^{T}\frac{B_{t+\tau-1,t+\tau}}{B_{t,t+T}}x_{t+T}\right].$$

$$b_{t,t+T} = \prod_{\tau=1}^{T-1}\frac{B_{t+\tau-1,t+\tau}}{B_{t,t+T}}.$$

$$E_t^Q(b_{t,t+T}) = 1$$

The Forward Price

$$F_{t,t+T} = E_t^Q(b_{t,t+T}x_{t+T}).$$

$$F_{t,t+T} = E_t^Q(b_{t,t+T})E_t^Q(x_{t+T}) + cov_t^Q(b_{t,t+T}, x_{t+T}).$$

$$F_{t,t+T} = H_{t,t+T} + cov_t^Q(b_{t,t+T}, x_{t+T}).$$

The Forward Price: Interpretation

- Forward price = Futures price + covariance term
- Covariance may be positive for many assets
- Covariance is positive for bonds
- For bonds, Forward price > Futures price

An Example: Lognormal Variables

Assume:

- x_{t+T} is lognormal
- $\phi^*_{t,t+T} = \phi_{t,t+1}\phi_{t+1,t+2}\phi_{t+2,t+3}\ldots$ is lognormal
- $b_{t,t+T}$ is lognormal

If x, y are lognormal with

$$\mu_x, \sigma_x$$

$$\mu_y, \sigma_y$$

$$\sigma_{x,y}$$

Then

$$E(xy) = E(x)E(y)e^{\sigma_{x,y}}$$

Futures price:

$$\boldsymbol{H_{t,t+T} = E(x\phi)}$$

$$\boldsymbol{= E(x)E(\phi)e^{\sigma_{x,\phi}}}$$

$$\boldsymbol{= E(x)e^{\sigma_{x,\phi}}}$$

$$\boldsymbol{= e^{\mu_x + \frac{1}{2}\sigma_x^2} e^{\sigma_{x,\phi}}}$$

$$\boldsymbol{= e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x,\phi}}}$$

Forward price

$$\boldsymbol{F_{t,t+T} = E(xb\phi)}$$

$$\boldsymbol{= E(x)E(b\phi)e^{cov(\ln x, \ln b\phi)}}$$

$$\boldsymbol{= E(x)e^{cov(\ln x, \ln b) + cov(\ln x, \ln \phi)}}$$

$$\boldsymbol{= e^{\mu_x + \frac{1}{2}\sigma_x^2} e^{\sigma_{x,b} + \sigma_{x,\phi}}}$$

$$\boldsymbol{= e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x,b} + \sigma_{x,\phi}}}$$

The Forward and Futures Prices: Lognormal Variables

Then:

- Futures price:

$$H_{t,t+T} = e^{\mu_x + \frac{1}{2}\sigma_x^2 + \sigma_{x\phi}}.$$

- Forward price

$$F_{t,t+T} = e^{\mu_x + \frac{1}{2}\sigma_x^2} e^{\sigma_{x\phi} + \sigma_{xb}}.$$

The Forward-Futures Bias: Lognormal Variables

- $$F_{t,t+T} = H_{t,t+T}\; e^{\sigma_{xb}}$$

- $$\sigma_{xb} = \sigma_x \sigma_b \rho_{xb}$$

- For Bonds
$$\rho_{xb} > 0$$

- σ is non-annualised

- Forward-futures bias increases with maturity of futures

The Forward-Futures Bias: Further Results

- Interest rates: inversely related to bond prices
- Section 6.5: define futures rate by:

$$H_{t,t+T} = e^{-h_{t,t+T}}.$$

- Define forward rate by:

$$F_{t,t+T} = e^{-f_{t,t+T}}.$$

-

$$\ln\left(\frac{F_{t,t+T}}{H_{t,t+T}}\right) = h_{t,t+T} - f_{t,t+T} = \sigma_{xb},$$

- Futures rate > Forward rate
- Important for interest-rate modelling (see ch 7)

The Forward-Futures Bias: Further Results

- For contingent claims (options) bias is magnified
- On LIFFE, Sydney, options traded on futures basis
- See section 6.6

The Forward-Futures Bias: Conclusions

- Forward-Futures bias depends on covariance of asset price with the bond roll-up factor
- Bias is more significant for bond futures
- Forward price $>$ Futures price
- Interest rates: Forward rate $<$ Futures rate
- Bias increases with maturity of futures contract
- Bias is magnified in the case of futures on options